**8.3 Analyzing Geometric Sequences and Series**

**Essential Question** How can you recognize a geometric sequence from its graph?

In a **geometric sequence**, the ratio of any term to the previous term, called the **common ratio**, is constant. For example, in the geometric sequence 1, 2, 4, 8, . . . , the common ratio is 2.

**EXPLORATION 1** Recognizing Graphs of Geometric Sequences

**Work with a partner.** Determine whether each graph shows a geometric sequence. If it does, then write a rule for the $n^{th}$ term of the sequence and use a spreadsheet to find the sum of the first 20 terms. What do you notice about the graph of a geometric sequence?

(a) ![Graph](image)

(b) ![Graph](image)

(c) ![Graph](image)

(d) ![Graph](image)

**EXPLORATION 2** Finding the Sum of a Geometric Sequence

**Work with a partner.** You can write the $n^{th}$ term of a geometric sequence with first term $a_1$ and common ratio $r$ as

$$a_n = a_1 r^{n-1}.$$  

So, you can write the sum $S_n$ of the first $n$ terms of a geometric sequence as

$$S_n = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^{n-1}.$$  

Rewrite this formula by finding the difference $S_n - rS_n$ and solving for $S_n$. Then verify your rewritten formula by finding the sums of the first 20 terms of the geometric sequences in Exploration 1. Compare your answers to those you obtained using a spreadsheet.

**Communicate Your Answer**

3. How can you recognize a geometric sequence from its graph?

4. Find the sum of the terms of each geometric sequence.
   
   a. 1, 2, 4, 8, . . . , 8192  
   b. 0.1, 0.01, 0.001, 0.0001, . . . , $10^{-10}$
What You Will Learn

- Identify geometric sequences.
- Write rules for geometric sequences.
- Find sums of finite geometric series.

Identifying Geometric Sequences

In a geometric sequence, the ratio of any term to the previous term is constant. This constant ratio is called the common ratio and is denoted by \( r \).

**Example 1** Identifying Geometric Sequences

Tell whether each sequence is geometric.

a. 6, 12, 20, 30, 42, . . .

b. 256, 64, 16, 4, 1, . . .

**Solution**

Find the ratios of consecutive terms.

\[
\begin{align*}
\frac{a_2}{a_1} &= \frac{12}{6} = 2 \\
\frac{a_3}{a_2} &= \frac{20}{12} = \frac{5}{3} \\
\frac{a_4}{a_3} &= \frac{30}{20} = \frac{3}{2} \\
\frac{a_5}{a_4} &= \frac{42}{30} = \frac{7}{5}
\end{align*}
\]

- The ratios are not constant, so the sequence is not geometric.

\[
\begin{align*}
\frac{a_2}{a_1} &= \frac{64}{256} = \frac{1}{4} \\
\frac{a_3}{a_2} &= \frac{16}{64} = \frac{1}{4} \\
\frac{a_4}{a_3} &= \frac{4}{16} = \frac{1}{4} \\
\frac{a_5}{a_4} &= \frac{1}{4}
\end{align*}
\]

- Each ratio is \( \frac{1}{4} \), so the sequence is geometric.

**Monitoring Progress**

Help in English and Spanish at [BigIdeasMath.com](http://BigIdeasMath.com)

Tell whether the sequence is geometric. Explain your reasoning.

1. 27, 9, 3, 1, \( \frac{1}{3} \), . . .

2. 2, 6, 24, 120, 720, . . .

3. \( -1, 2, -4, 8, -16, \ldots \)

**Writing Rules for Geometric Sequences**

**Core Concept**

**Rule for a Geometric Sequence**

**Algebra** The \( n \)th term of a geometric sequence with first term \( a_1 \) and common ratio \( r \) is given by:

\[
a_n = a_1 r^{n-1}
\]

**Example** The \( n \)th term of a geometric sequence with a first term of 2 and a common ratio of 3 is given by:

\[
a_n = 2(3)^{n-1}
\]
Section 8.3  Analyzing Geometric Sequences and Series

Writing a Rule for the \( n \)th Term

Write a rule for the \( n \)th term of each sequence. Then find \( a_8 \).

a. 5, 15, 45, 135, . . .

b. 88, –44, 22, –11, . . .

**SOLUTION**

a. The sequence is geometric with first term \( a_1 = 5 \) and common ratio \( r = \frac{15}{5} = 3 \).

So, a rule for the \( n \)th term is

\[
    a_n = a_1 r^{n-1}.
\]

Substitute 5 for \( a_1 \) and 3 for \( r \).

\[
    a_8 = 5(3)^{8-1} = 5(3)^7 = 10,935.
\]

b. The sequence is geometric with first term \( a_1 = 88 \) and common ratio \( r = \frac{-44}{88} = -\frac{1}{2} \).

So, a rule for the \( n \)th term is

\[
    a_n = a_1 r^{n-1}.
\]

Substitute 88 for \( a_1 \) and \(-\frac{1}{2}\) for \( r \).

\[
    a_8 = 88 \left(-\frac{1}{2}\right)^{8-1} = -\frac{11}{16}.
\]

Monitoring Progress

**EXAMPLE 2** Writing a Rule for the \( n \)th Term

Write a rule for the \( n \)th term of each sequence. Then find \( a_9 \).

a. 5, 15, 45, 135, . . .

b. 88, –44, 22, –11, . . .

**SOLUTION**

a. The sequence is geometric with first term \( a_1 = 5 \) and common ratio \( r = \frac{15}{5} = 3 \).

So, a rule for the \( n \)th term is

\[
    a_n = a_1 r^{n-1}.
\]

Substitute 5 for \( a_1 \) and 3 for \( r \).

\[
    a_8 = 5(3)^{8-1} = 5(3)^7 = 10,935.
\]

b. The sequence is geometric with first term \( a_1 = 88 \) and common ratio \( r = \frac{-44}{88} = -\frac{1}{2} \).

So, a rule for the \( n \)th term is

\[
    a_n = a_1 r^{n-1}.
\]

Substitute 88 for \( a_1 \) and \(-\frac{1}{2}\) for \( r \).

\[
    a_8 = 88 \left(-\frac{1}{2}\right)^{8-1} = -\frac{11}{16}.
\]

**EXAMPLE 3** Writing a Rule Given a Term and Common Ratio

One term of a geometric sequence is \( a_4 = 12 \). The common ratio is \( r = 2 \). Write a rule for the \( n \)th term. Then graph the first six terms of the sequence.

**SOLUTION**

Step 1

Use the general rule to find the first term.

\[
    a_n = a_1 r^{n-1}.
\]

Substitute 4 for \( n \).

\[
    12 = a_1(2)^3 \quad \text{Substitute 12 for } a_4 \text{ and } 2 \text{ for } r.
\]

Solve for \( a_1 \).

\[
    1.5 = a_1.
\]

Step 2

Write a rule for the \( n \)th term.

\[
    a_n = a_1 r^{n-1}.
\]

Substitute 1.5 for \( a_1 \) and 2 for \( r \).

\[
    a_n = 1.5(2)^{n-1}.
\]

Step 3

Use the rule to create a table of values for the sequence. Then plot the points.

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_n )</td>
<td>1.5</td>
<td>3</td>
<td>6</td>
<td>12</td>
<td>24</td>
<td>48</td>
</tr>
</tbody>
</table>

Monitoring Progress

**EXAMPLE 3** Writing a Rule Given a Term and Common Ratio

One term of a geometric sequence is \( a_4 = 12 \). The common ratio is \( r = 2 \). Write a rule for the \( n \)th term. Then graph the first six terms of the sequence.

**SOLUTION**

Step 1

Use the general rule to find the first term.

\[
    a_n = a_1 r^{n-1}.
\]

Substitute 4 for \( n \).

\[
    12 = a_1(2)^3 \quad \text{Substitute 12 for } a_4 \text{ and } 2 \text{ for } r.
\]

Solve for \( a_1 \).

\[
    1.5 = a_1.
\]

Step 2

Write a rule for the \( n \)th term.

\[
    a_n = a_1 r^{n-1}.
\]

Substitute 1.5 for \( a_1 \) and 2 for \( r \).

\[
    a_n = 1.5(2)^{n-1}.
\]

Step 3

Use the rule to create a table of values for the sequence. Then plot the points.

<table>
<thead>
<tr>
<th>( n )</th>
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<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_n )</td>
<td>1.5</td>
<td>3</td>
<td>6</td>
<td>12</td>
<td>24</td>
<td>48</td>
</tr>
</tbody>
</table>

Analyzing Relationships

Notice that the points lie on an exponential curve because consecutive terms change by equal factors. So, a geometric sequence in which \( r > 0 \) and \( r \neq 1 \) is an exponential function whose domain is a subset of the integers.
**Example 4** Writing a Rule Given Two Terms

Two terms of a geometric sequence are \(a_2 = 12\) and \(a_5 = -768\). Write a rule for the \(n\)th term.

**Solution**

Step 1 Write a system of equations using \(a_n = a_1 r^{n-1}\). Substitute 2 for \(n\) to write Equation 1. Substitute 5 for \(n\) to write Equation 2.

\[
\begin{align*}
a_2 &= a_1 r^{2-1} \\ a_5 &= a_1 r^{5-1}
\end{align*}
\]

\[
\begin{align*}
12 &= a_1 r \\ -768 &= a_1 r^4
\end{align*}
\]

Equation 1

Equation 2

Step 2 Solve the system.

\[
\begin{align*}
r &= \frac{a_1}{12} \\
-768 &= \frac{12}{r} (r^4)
\end{align*}
\]

Substitute for \(a_1\) in Equation 2.

\[
-768 = 12 \cdot r^3
\]

Simplify.

\[
-4 = r
\]

Solve for \(r\).

\[
a_1 = \frac{12}{r^4}
\]

Substitute for \(r\) in Equation 1.

\[
-3 = a_1
\]

Solve for \(a_1\).

Step 3 Write a rule for \(a_n\).

\[
a_n = a_1 r^{n-1}
\]

Write general rule.

\[
= -3(-4)^{n-1}
\]

Substitute for \(a_1\) and \(r\).

**Monitoring Progress**

Write a rule for the \(n\)th term of the sequence. Then graph the first six terms of the sequence.

5. \(a_6 = -96, \ r = -2\)

6. \(a_2 = 12, \ a_4 = 3\)

**Finding Sums of Finite Geometric Series**

The expression formed by adding the terms of a geometric sequence is called a geometric series. The sum of the first \(n\) terms of a geometric series is denoted by \(S_n\).

You can develop a rule for \(S_n\) as follows.

\[
S_n = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^{n-1}
\]

\[
-rS_n = \quad -a_1 r - a_1 r^2 - a_1 r^3 - \cdots - a_1 r^{n-1} - a_1 r^n
\]

\[
S_n - rS_n = a_1 + 0 + 0 + \cdots + 0 - a_1 r^n
\]

\[
S_n(1 - r) = a_1(1 - r^n)
\]

When \(r \neq 1\), you can divide each side of this equation by \(1 - r\) to obtain the following rule for \(S_n\).

**Core Concept**

The Sum of a Finite Geometric Series

The sum of the first \(n\) terms of a geometric series with common ratio \(r \neq 1\) is

\[
S_n = a_1 \left(\frac{1 - r^n}{1 - r}\right).
\]
Section 8.3  Analyzing Geometric Sequences and Series

### Example 5: Finding the Sum of a Geometric Series

Find the sum \( \sum_{k=1}^{10} 4(3)^{k-1} \).

**SOLUTION**

**Step 1**  Find the first term and the common ratio.
- \( a_1 = 4(3)^{1-1} = 4 \)  
- \( r = 3 \)

**Step 2**  Find the sum.
- \( S_{10} = a_1 \left( \frac{1 - r^{10}}{1 - r} \right) \)
- \( = 4 \left( \frac{1 - 3^{10}}{1 - 3} \right) \)
- \( = 118,096 \)  

### Example 6: Solving a Real-Life Problem

You can calculate the monthly payment \( M \) (in dollars) for a loan using the formula

\[
M = \frac{L}{\sum_{k=1}^{t} \left( \frac{1}{1 + i} \right)^k}
\]

where \( L \) is the loan amount (in dollars), \( i \) is the monthly interest rate (in decimal form), and \( t \) is the term (in months). Calculate the monthly payment on a 5-year loan for $20,000 with an annual interest rate of 6%.

**SOLUTION**

**Step 1**  Substitute for \( L \), \( i \), and \( t \). The loan amount is \( L = 20,000 \), the monthly interest rate is \( i = \frac{0.06}{12} = 0.005 \), and the term is \( t = 5(12) = 60 \).

**Step 2**  Notice that the denominator is a geometric series with first term \( \frac{1}{1.005^k} \) and common ratio \( \frac{1}{1.005} \). Use a calculator to find the monthly payment.

So, the monthly payment is $386.66.

### Monitoring Progress

Find the sum.

7. \( \sum_{k=1}^{8} 5^k - 1 \)
8. \( \sum_{i=1}^{12} (-2)^{i-1} \)
9. \( \sum_{r=1}^{7} -16(0.5)^{r-1} \)

10. **WHAT IF?** In Example 6, how does the monthly payment change when the annual interest rate is 5%?
Monitoring Progress and Modeling with Mathematics

In Exercises 5–12, tell whether the sequence is geometric. Explain your reasoning. (See Example 1.)

5. 96, 48, 24, 12, 6, . . .
6. 729, 243, 81, 27, 9, . . .
7. 2, 4, 6, 8, 10, . . .
8. 5, 20, 35, 50, 65, . . .
9. 0.2, 3.2, −12.8, 51.2, −204.8, . . .
10. 0.3, −1.5, 7.5, −37.5, 187.5, . . .
11. \(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots\)
12. \(1, \frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \frac{1}{256}, \frac{1}{1024}, \ldots\)

13. WRITING EQUATIONS Write a rule for the geometric sequence with the given description.
   a. The first term is −3, and each term is 5 times the previous term.
   b. The first term is 72, and each term is \(\frac{1}{2}\) times the previous term.

14. WRITING Compare the terms of a geometric sequence when \(r > 1\) to when \(0 < r < 1\).

In Exercises 15–22, write a rule for the \(n\)th term of the sequence. Then find \(a_9\). (See Example 2.)

15. 4, 20, 100, 500, . . .
16. 6, 24, 96, 384, . . .
17. 112, 56, 28, 14, . . .
18. 375, 75, 15, 3, . . .
19. 4, 6, \(\frac{27}{2}\), . . .
20. 2, \(\frac{9}{2}\), \(\frac{27}{8}\), . . .
21. 1.3, −3.9, 11.7, −35.1, . . .
22. 1.5, −7.5, 37.5, −187.5, . . .

ERROR ANALYSIS In Exercises 31 and 32, describe and correct the error in writing a rule for the \(n\)th term of the geometric sequence for which \(a_2 = 48\) and \(r = 6\).

\[
\begin{align*}
\text{31.} & \quad a_n = a_1 r^n \\
& \quad 48 = a_1 6^2 \\
& \quad a_1 = 4 \quad \text{incorrect} \\
& \quad a_n = 4 \left(\frac{1}{3}\right) (6)^n \\
\end{align*}
\]

\[
\begin{align*}
\text{32.} & \quad a_n = r (a_1)^{n-1} \\
& \quad 48 = 6(a_2)^{2-1} \\
& \quad 6 = a_1 \\
& \quad a_n = 6(6)^{n-1} \\
\end{align*}
\]

In Exercises 33–40, write a rule for the \(n\)th term of the geometric sequence. (See Example 4.)

33. \(a_2 = 28, a_5 = 1792\)
34. \(a_1 = 11, a_4 = 88\)
35. \(a_1 = −6, a_5 = −486\)
36. \(a_2 = −10, a_6 = −6250\)
37. \(a_2 = 64, a_4 = 1\)
38. \(a_1 = 1, a_2 = 49\)
39. \(a_2 = −72, a_6 = −\frac{1}{18}\)
40. \(a_2 = −48, a_5 = \frac{3}{4}\)
WRITING EQUATIONS In Exercises 41–46, write a rule for the sequence with the given terms.

41. \( a_n = 2^n \)  
   \( (4, 32) \)  
   \( (2, 8) \)  
   \( (1, 4) \)

42. \( a_n = 2^n \)  
   \( (4, 135) \)  
   \( (2, 12) \)  
   \( (3, 3) \)

43. \( a_n = 2^n \)  
   \( (1, 5) \)  
   \( (2, 2.5) \)  
   \( (4, 0.625) \)

44. \( a_n = 2^n \)  
   \( (3, 1.25) \)  
   \( (4, 0.625) \)  
   \( (2, 2.5) \)

45. \( a_n = 2^n \)  
   \( (3, 16) \)  
   \( (4, 32) \)  
   \( (2, 8) \)  
   \( (1, 4) \)

46. \( a_n = 2^n \)  
   \( (3, 45) \)  
   \( (4, 135) \)  
   \( (2, 15) \)  
   \( (1, 5) \)

In Exercises 47–52, find the sum. (See Example 5.)

47. \( \sum_{i=1}^{10} 6(7)^{i-1} \)

48. \( \sum_{i=1}^{10} 7(4)^{i-1} \)

49. \( \sum_{i=1}^{10} \left( \frac{3}{4} \right)^{i-1} \)

50. \( \sum_{i=1}^{10} \left( \frac{1}{3} \right)^{i-1} \)

51. \( \sum_{i=0}^{8} 8 \left( \frac{2}{3} \right)^i \)

52. \( \sum_{i=0}^{9} 9 \left( \frac{3}{4} \right)^i \)

NUMBER SENSE In Exercises 53 and 54, find the sum.

53. The first 8 terms of the geometric sequence: 
   \( -12, -48, -192, -768, \ldots \)

54. The first 9 terms of the geometric sequence: 
   \( -14, -42, -126, -378, \ldots \)

55. WRITING Compare the graph of \( a_n = 5(3)^n - 1 \), where \( n \) is a positive integer, to the graph of \( f(x) = 5 \cdot 3^x - 1 \), where \( x \) is a real number.

56. ABSTRACT REASONING Use the rule for the sum of a finite geometric series to write each polynomial as a rational expression.
   a. \( 1 + x + x^2 + x^3 + x^4 \)
   b. \( 3x + 6x^3 + 12x^5 + 24x^7 \)

MODELING WITH MATHEMATICS In Exercises 57 and 58, use the monthly payment formula given in Example 6.

57. You are buying a new car. You take out a 5-year loan for $15,000. The annual interest rate of the loan is 4%. Calculate the monthly payment. (See Example 6.)

58. You are buying a new house. You take out a 30-year mortgage for $200,000. The annual interest rate of the loan is 4.5%. Calculate the monthly payment.

59. MODELING WITH MATHEMATICS A regional soccer tournament has 64 participating teams. In the first round of the tournament, 32 games are played. In each successive round, the number of games decreases by a factor of \( \frac{1}{2} \).
   a. Write a rule for the number of games played in the \( n \)th round. For what values of \( n \) does the rule make sense? Explain.
   b. Find the total number of games played in the regional soccer tournament.

60. MODELING WITH MATHEMATICS In a skydiving formation with \( R \) rings, each ring after the first has twice as many skydivers as the preceding ring. The formation for \( R = 2 \) is shown.

   a. Let \( a_n \) be the number of skydivers in the \( n \)th ring. Write a rule for \( a_n \).
   b. Find the total number of skydivers when there are four rings.
61. **PROBLEM SOLVING** The Sierpinski carpet is a fractal created using squares. The process involves removing smaller squares from larger squares. First, divide a large square into nine congruent squares. Then remove the center square. Repeat these steps for each smaller square, as shown below. Assume that each side of the initial square is 1 unit long.

![Stage 1, Stage 2, Stage 3 images]

**a.** Let \(a_n\) be the total number of squares removed at the \(n\)th stage. Write a rule for \(a_n\). Then find the total number of squares removed through Stage 8.

**b.** Let \(b_n\) be the remaining area of the original square after the \(n\)th stage. Write a rule for \(b_n\). Then find the remaining area of the original square after Stage 12.

62. **HOW DO YOU SEE IT?** Match each sequence with its graph. Explain your reasoning.

- **a.** \(a_n = 10 \left( \frac{1}{2} \right)^{n-1} \)  
  - **b.** \(a_n = 10(2)^{n-1} \)

![Graph images A and B]

63. **CRITICAL THINKING** On January 1, you deposit $2000 in a retirement account that pays 5% annual interest. You make this deposit each January 1 for the next 30 years. How much money do you have in your account immediately after you make your last deposit?

64. **THOUGHT PROVOKING** The first four iterations of the fractal called the Koch snowflake are shown below. Find the perimeter and area of each iteration. Do the perimeters and areas form geometric sequences? Explain your reasoning.

![Koch snowflakes images]

65. **MAKING AN ARGUMENT** You and your friend are comparing two loan options for a $165,000 house. Loan 1 is a 15-year loan with an annual interest rate of 3%. Loan 2 is a 30-year loan with an annual interest rate of 4%. Your friend claims the total amount repaid over the loan will be less for Loan 2. Is your friend correct? Justify your answer.

66. **CRITICAL THINKING** Let \(L\) be the amount of a loan (in dollars), \(i\) be the monthly interest rate (in decimal form), \(t\) be the term (in months), and \(M\) be the monthly payment (in dollars).

**a.** When making monthly payments, you are paying the loan amount plus the interest the loan gathers each month. For a 1-month loan, \(t = 1\), the equation for repayment is \(L(1 + i) - M = 0\). For a 2-month loan, \(t = 2\), the equation is \([L(1 + i) - M](1 + i) - M = 0\). Solve both of these repayment equations for \(L\).

**b.** Use the pattern in the equations you solved in part (a) to write a repayment equation for a \(t\)-month loan. (Hint: \(L\) is equal to \(M\) times a geometric series.) Then solve the equation for \(M\).

**c.** Use the rule for the sum of a finite geometric series to show that the formula in part (b) is equivalent to

\[
M = L \left( \frac{i}{1 - (1 + i)^{-t}} \right)
\]

Use this formula to check your answers in Exercises 57 and 58.

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**Maintaining Mathematical Proficiency**

Reviewing what you learned in previous grades and lessons

**Graph the function. State the domain and range.**  
*(Section 7.2)*

67. \(f(x) = \frac{1}{x - 3}\) 
68. \(g(x) = \frac{2}{x} + 3\)

69. \(h(x) = \frac{1}{x - 2} + 1\) 
70. \(p(x) = \frac{3}{x + 1} - 2\)