8.5 Using Intercept Form

Essential Question: What are some of the characteristics of the graph of \( f(x) = a(x - p)(x - q) \)?

Exploration 1: Using Zeros to Write Functions

Work with a partner. Each graph represents a function of the form \( f(x) = (x - p)(x - q) \) or \( f(x) = -(x - p)(x - q) \). Write the function represented by each graph. Explain your reasoning.

Communicate Your Answer

2. What are some of the characteristics of the graph of \( f(x) = a(x - p)(x - q) \)?

3. Consider the graph of \( f(x) = a(x - p)(x - q) \).
   a. Does changing the sign of \( a \) change the \( x \)-intercepts? Does changing the sign of \( a \) change the \( y \)-intercept? Explain your reasoning.
   b. Does changing the value of \( p \) change the \( x \)-intercepts? Does changing the value of \( p \) change the \( y \)-intercept? Explain your reasoning.
What You Will Learn

- Graph quadratic functions of the form \( f(x) = a(x - p)(x - q) \).
- Use intercept form to find zeros of functions.
- Use characteristics to graph and write quadratic functions.
- Use characteristics to graph and write cubic functions.

Graphing \( f(x) = a(x - p)(x - q) \)

You have already graphed quadratic functions written in several different forms, such as \( f(x) = ax^2 + bx + c \) (standard form) and \( g(x) = a(x - h)^2 + k \) (vertex form).

Quadratic functions can also be written in intercept form, \( f(x) = a(x - p)(x - q) \), where \( a \neq 0 \). In this form, the polynomial that defines a function is in factored form and the \( x \)-intercepts of the graph can be easily determined.

Core Concept

Graphing \( f(x) = a(x - p)(x - q) \)

- The \( x \)-intercepts are \( p \) and \( q \).
- The axis of symmetry is halfway between \((p, 0)\) and \((q, 0)\). So, the axis of symmetry is \( x = \frac{p + q}{2} \).
- The graph opens up when \( a > 0 \), and the graph opens down when \( a < 0 \).

Example 1: Graphing \( f(x) = a(x - p)(x - q) \)

Graph \( f(x) = -(x + 1)(x - 5) \). Describe the domain and range.

SOLUTION

Step 1 Identify the \( x \)-intercepts. Because the \( x \)-intercepts are \( p = -1 \) and \( q = 5 \), plot \((-1, 0)\) and \((5, 0)\).

Step 2 Find and graph the axis of symmetry.

\[
x = \frac{p + q}{2} = \frac{-1 + 5}{2} = 2
\]

Step 3 Find and plot the vertex.

The \( x \)-coordinate of the vertex is 2.
To find the \( y \)-coordinate of the vertex, substitute 2 for \( x \) and simplify.

\[
f(2) = -(2 + 1)(2 - 5) = 9
\]

So, the vertex is \((2, 9)\).

Step 4 Draw a parabola through the vertex and the points where the \( x \)-intercepts occur.

- The domain is all real numbers. The range is \( y \leq 9 \).
**Example 2**  
**Graphing a Quadratic Function**

Graph \( f(x) = 2x^2 - 8 \). Describe the domain and range.

**Solution**

**Step 1**  
Rewrite the quadratic function in intercept form.

\[
\begin{align*}
 f(x) &= 2x^2 - 8 \\
 &= 2(x^2 - 4) \\
 &= 2(x + 2)(x - 2)
\end{align*}
\]

**Step 2**  
Identify the \( x \)-intercepts. Because the \( x \)-intercepts are \( p = -2 \) and \( q = 2 \), plot \((-2, 0)\) and \((2, 0)\).

**Step 3**  
Find and graph the axis of symmetry.\[
\begin{align*}
 x &= \frac{p + q}{2} = \frac{-2 + 2}{2} = 0
\end{align*}
\]

**Step 4**  
Find and plot the vertex.

The \( x \)-coordinate of the vertex is 0.  
The \( y \)-coordinate of the vertex is \( f(0) = 2(0)^2 - 8 = -8 \).

So, the vertex is \((0, -8)\).

**Step 5**  
Draw a parabola through the vertex and the points where the \( x \)-intercepts occur.

- The domain is all real numbers. The range is \( y \geq -8 \).

**Monitoring Progress**

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Graph the quadratic function. Label the vertex, axis of symmetry, and \( x \)-intercepts. Describe the domain and range of the function.

1. \( f(x) = (x + 2)(x - 3) \)
2. \( g(x) = -2(x - 4)(x + 1) \)
3. \( h(x) = 4x^2 - 36 \)

**Remember**

Functions have zeros, and graphs have \( x \)-intercepts.

**Using Intercept Form to Find Zeros of Functions**

In Section 8.2, you learned that a zero of a function is an \( x \)-value for which \( f(x) = 0 \).  
You can use the intercept form of a function to find the zeros of the function.

**Example 3**  
**Finding Zeros of a Function**

Find the zeros of \( f(x) = (x - 1)(x + 2) \).

**Solution**

To find the zeros, determine the \( x \)-values for which \( f(x) = 0 \).

\[
\begin{align*}
 f(x) &= (x - 1)(x + 2) \\
 0 &= (x - 1)(x + 2) \\
 x - 1 &= 0 & \text{or} & & x + 2 &= 0 \quad \text{Zero-Product Property} \\
 x &= 1 & \text{or} & & x &= -2 \quad \text{Solve for } x.
\end{align*}
\]

So, the zeros of the function are \(-2\) and \(1\).
**Core Concept**

**Factors and Zeros**
For any factor \(x - n\) of a polynomial, \(n\) is a zero of the function defined by the polynomial.

**Example 4** Finding Zeros of Functions

Find the zeros of each function.

a. \(f(x) = -2x^2 - 10x - 12\)

b. \(h(x) = (x - 1)(x^2 - 16)\)

**Solution**

Write each function in intercept form to identify the zeros.

a. \(f(x) = -2x^2 - 10x - 12\)

Write the function.

\[
= -2(x^2 + 5x + 6)
\]

Factor out common factor.

\[
= -2(x + 3)(x + 2)
\]

Factor the trinomial.

\[
\text{So, the zeros of the function are } -3 \text{ and } -2.
\]

b. \(h(x) = (x - 1)(x^2 - 16)\)

Write the function.

\[
= (x - 1)(x + 4)(x - 4)
\]

Difference of two squares pattern

\[
\text{So, the zeros of the function are } -4, 1, \text{ and } 4.
\]

**Monitoring Progress**

Find the zero(s) of the function.

4. \(f(x) = (x - 6)(x - 1)\)

5. \(g(x) = 3x^2 - 12x + 12\)

6. \(h(x) = x(x^2 - 1)\)

**Using Characteristics to Graph and Write Quadratic Functions**

**Example 5** Graphing a Quadratic Function Using Zeros

Use zeros to graph \(h(x) = x^2 - 2x - 3\).

**Solution**

The function is in standard form. You know that the parabola opens up \((a > 0)\) and the \(y\)-intercept is \(-3\). So, begin by plotting \((0, -3)\).

Notice that the polynomial that defines the function is factorable. So, write the function in intercept form and identify the zeros.

\[
h(x) = x^2 - 2x - 3
\]

Write the function.

\[
= (x + 1)(x - 3)
\]

Factor the trinomial.

The zeros of the function are \(-1\) and \(3\). So, plot \((-1, 0)\) and \((3, 0)\). Draw a parabola through the points.
Writing Quadratic Functions

Write a quadratic function in standard form whose graph satisfies the given condition(s).

a. vertex: \((-3, 4)\)

b. passes through \((-9, 0), (-2, 0),\) and \((-4, 20)\)

**SOLUTION**

**a.** Because you know the vertex, use vertex form to write a function.

\[
f(x) = a(x - h)^2 + k
\]

\[
= a(x + 3)^2 + 4
\]

\[
= x^2 + 6x + 9 + 4
\]

\[
= x^2 + 6x + 13
\]

Substitute for \(a, h,\) and \(k.\)

**b.** The given points indicate that the \(x\)-intercepts are \(-9\) and \(-2.\) So, use intercept form to write a function.

\[
f(x) = a(x - p)(x - q)
\]

\[
= a(x + 9)(x + 2)
\]

Use the other given point, \((-4, 20),\) to find the value of \(a.\)

\[
20 = a(-4 + 9)(-4 + 2)
\]

\[
20 = a(5)(-2)
\]

\[
-2 = a
\]

Simplify.

Use the value of \(a\) to write the function.

\[
-2(x + 9)(x + 2)
\]

\[
-2x^2 - 22x - 36
\]

Simplify.

**Monitoring Progress**

Use zeros to graph the function.

7. \(f(x) = (x - 1)(x - 4)\)

8. \(g(x) = x^2 + x - 12\)

Write a quadratic function in standard form whose graph satisfies the given condition(s).

9. \(x\)-intercepts: \(-1\) and \(1\)

10. \(x\)-intercepts: \((-8, 8)\)

11. passes through \((0, 0), (10, 0),\) and \((4, 12)\)

12. passes through \((-5, 0), (4, 0),\) and \((3, -16)\)

Using Characteristics to Graph and Write Cubic Functions

In Example 4, you extended the concept of intercept form to cubic functions.

\[
f(x) = a(x - p)(x - q)(x - r), a \neq 0
\]

Intercept form of a cubic function

The \(x\)-intercepts of the graph of \(f\) are \(p, q,\) and \(r.\)
EXAMPLE 7  Graphing a Cubic Function Using Zeros

Use zeros to graph \( f(x) = x^3 - 4x \).

**SOLUTION**

Notice that the polynomial that defines the function is factorable. So, write the function in intercept form and identify the zeros.

\[
 f(x) = x^3 - 4x \\
= x(x^2 - 4) \\
= x(x + 2)(x - 2)
\]

The zeros of the function are \(-2, 0, \) and \(2\). So, plot \((-2, 0), (0, 0), \) and \((2, 0)\).

To help determine the shape of the graph, find points between the zeros.

| \( x \) | \(-1\) | 1 |
| \( f(x) \) | 3 | \(-3\) |

Plot \((-1, 3)\) and \((1, -3)\). Draw a smooth curve through the points.

EXAMPLE 8  Writing a Cubic Function

The graph represents a cubic function. Write the function.

**SOLUTION**

From the graph, you can see that the \(x\)-intercepts are \(0, 2, \) and \(5\). Use intercept form to write a function.

\[
 f(x) = a(x - p)(x - q)(x - r) \\
= a(x - 0)(x - 2)(x - 5) \\
= a(x)(x - 2)(x - 5)
\]

Simplify.

Use the other given point, \((3, 12)\), to find the value of \(a\).

\[
12 = a(3)(3 - 2)(3 - 5) \\
-2 = a \\
Solve for \(a\).
\]

Use the value of \(a\) to write the function.

\[
 f(x) = -2(x)(x - 2)(x - 5) \\
= -2x^3 + 14x^2 - 20x
\]

Simplify.

The function represented by the graph is \( f(x) = -2x^3 + 14x^2 - 20x \).

**Monitoring Progress**

Use zeros to graph the function.

13. \( g(x) = (x - 1)(x - 3)(x + 3) \)
14. \( h(x) = x^3 - 6x^2 + 5x \)
15. The zeros of a cubic function are \(-3, -1, \) and \(1\). The graph of the function passes through the point \((0, -3)\). Write the function.
8.5 Exercises

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** The values $p$ and $q$ are _________ of the graph of the function $f(x) = a(x - p)(x - q)$.

2. **WRITING** Explain how to find the maximum value or minimum value of a quadratic function when the function is given in intercept form.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, find the $x$-intercepts and axis of symmetry of the graph of the function.

3. $y = (x - 1)(x + 3)$
4. $y = -2(x - 2)(x - 5)$
5. $f(x) = -5(x + 7)(x - 5)$
6. $g(x) = \frac{2}{3}x(x + 8)$

In Exercises 7–12, graph the quadratic function. Label the vertex, axis of symmetry, and $x$-intercepts. Describe the domain and range of the function. (See Example 1.)

7. $f(x) = (x + 4)(x + 1)$
8. $y = (x - 2)(x + 2)$
9. $y = -(x + 6)(x - 4)$
10. $h(x) = -4(x - 7)(x - 3)$
11. $g(x) = 5(x + 1)(x + 2)$
12. $y = -2(x - 3)(x + 4)$

In Exercises 13–20, graph the quadratic function. Label the vertex, axis of symmetry, and $x$-intercepts. Describe the domain and range of the function. (See Example 2.)

13. $y = x^2 - 9$
14. $f(x) = x^2 - 8x$
15. $h(x) = -5x^2 + 5x$
16. $y = 3x^2 - 48$
17. $g(x) = x^2 + 9x + 14$
18. $p(x) = x^2 + 6x - 27$
19. $y = 4x^2 - 30x + 32$
20. $y = -2x^2 - 4x + 30$

In Exercises 21–30, find the zero(s) of the function. (See Examples 3 and 4.)

21. $y = -2(x - 2)(x - 10)$
22. $f(x) = \frac{1}{3}(x + 5)(x - 1)$
23. $g(x) = x^2 + 5x - 24$
24. $y = x^2 - 17x + 52$
25. $y = 3x^2 - 15x - 42$
26. $g(x) = -4x^2 - 8x - 4$
27. $f(x) = (x + 5)(x^2 - 4)$
28. $h(x) = (x^2 - 36)(x - 11)$
29. $y = x^3 - 49x$
30. $y = x^3 - x^2 - 9x + 9$

In Exercises 31–36, match the function with its graph.

31. $y = (x + 5)(x + 3)$
32. $y = (x + 5)(x - 3)$
33. $y = (x - 5)(x + 3)$
34. $y = (x - 5)(x - 3)$
35. $y = (x + 5)(x - 5)$
36. $y = (x + 3)(x - 3)$

Section 8.5 Using Intercept Form
In Exercises 37–42, use zeros to graph the function. (See Example 5.)

37. \( f(x) = (x + 2)(x - 6) \) 38. \( g(x) = -3(x + 1)(x + 7) \)
39. \( y = x^2 - 11x + 18 \) 40. \( y = x^2 - x - 30 \)
41. \( y = -5x^2 - 10x + 40 \) 42. \( h(x) = 8x^2 - 8 \)

ERROR ANALYSIS In Exercises 43 and 44, describe and correct the error in finding the zeros of the function.

43. \( y = 5(x^2 + 3)(x - 2) \)
   The zeros of the function are \( 3 \) and \( -2 \).

44. \( y = (x + 4)(x^2 - 9) \)
   The zeros of the function are \( -4 \) and \( 9 \).

In Exercises 45–56, write a quadratic function in standard form whose graph satisfies the given condition(s). (See Example 6.)

45. vertex: \((7, -3)\) 46. vertex: \((4, 8)\)
47. \( x \)-intercepts: 1 and 9 48. \( x \)-intercepts: \(-2\) and \(-5\)
49. passes through \((-4, 0)\), \((3, 0)\), and \((2, -18)\)
50. passes through \((-5, 0)\), \((-1, 0)\), and \((-4, 3)\)
51. passes through \((7, 0)\)
52. passes through \((0, 0)\) and \((6, 0)\)
53. axis of symmetry: \(x = -5\)
54. \( y \) increases as \( x \) increases when \( x < 4 \); \( y \) decreases as \( x \) increases when \( x > 4 \).
55. range: \( y \geq -3 \) 56. range: \( y \leq 10 \)

In Exercises 57–60, write the quadratic function represented by the graph.

57. \[ \begin{array}{c}
\begin{array}{c}
\text{4.} \\
\text{2.} \\
\text{1.} \\
\text{3.} \\
\text{6.} \\
\text{5.} \\
\text{4.} \\
\text{3.} \\
\text{2.} \\
\text{1.} \\
\text{0.} \\
\text{-1.} \\
\text{-2.} \\
\text{-3.} \\
\end{array}
\end{array} \]
58. \[ \begin{array}{c}
\begin{array}{c}
\text{-2.} \\
\text{-1.} \\
\text{0.} \\
\text{1.} \\
\text{2.} \\
\text{3.} \\
\text{4.} \\
\text{5.} \\
\end{array}
\end{array} \]

59. \[ \begin{array}{c}
\begin{array}{c}
\text{35} \\
\text{30} \\
\text{25} \\
\text{20} \\
\text{15} \\
\text{10} \\
\text{5} \\
\text{0} \\
\end{array}
\end{array} \]
60. \[ \begin{array}{c}
\begin{array}{c}
\text{36} \\
\text{30} \\
\text{24} \\
\text{18} \\
\text{12} \\
\text{6} \\
\text{0} \\
\end{array}
\end{array} \]

In Exercises 61–68, use zeros to graph the function. (See Example 7.)

61. \( y = 5x(x + 2)(x - 6) \) 62. \( f(x) = -(x + 9)(x + 3) \)
63. \( h(x) = (x - 2)(x + 2)(x + 7) \) 64. \( y = (x + 1)(x - 5)(x - 4) \)
65. \( f(x) = 3x^3 - 48x \) 66. \( y = -2x^3 + 20x^2 - 50x \)
67. \( y = -x^3 - 16x^2 - 28x \) 68. \( g(x) = 6x^3 + 30x^2 - 36x \)

In Exercises 69–72, write the cubic function represented by the graph. (See Example 8.)

69. \[ \begin{array}{c}
\begin{array}{c}
\text{-10} \\
\text{-8} \\
\text{-6} \\
\text{-4} \\
\text{-2} \\
\text{0} \\
\text{2} \\
\text{4} \\
\text{6} \\
\end{array}
\end{array} \]
70. \[ \begin{array}{c}
\begin{array}{c}
\text{-20} \\
\text{-16} \\
\text{-12} \\
\text{-8} \\
\text{-4} \\
\text{0} \\
\text{4} \\
\text{8} \\
\end{array}
\end{array} \]
71. \[ \begin{array}{c}
\begin{array}{c}
\text{-2} \\
\text{0} \\
\text{2} \\
\text{4} \\
\text{6} \\
\text{8} \\
\text{10} \\
\text{12} \\
\end{array}
\end{array} \]
72. \[ \begin{array}{c}
\begin{array}{c}
\text{-2} \\
\text{-4} \\
\text{-6} \\
\text{-8} \\
\text{-10} \\
\text{-12} \\
\text{-14} \\
\text{-16} \\
\text{-18} \\
\end{array}
\end{array} \]

In Exercises 73–76, write a cubic function whose graph satisfies the given condition(s).

73. \( x \)-intercepts: \(-2\), 3, and 8
74. \( x \)-intercepts: \(-7\), \(-5\), and 0
75. passes through \((1, 0)\) and \((7, 0)\)
76. passes through \((0, 6)\)
In Exercises 77–80, all the zeros of a function are given. Use the zeros and the other point given to write a quadratic or cubic function represented by the table.

**77.**

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>30</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
</tr>
</tbody>
</table>

**Exercise 77 Answer:**

**78.**

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>−3</td>
<td>0</td>
</tr>
<tr>
<td>−1</td>
<td>−72</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
</tr>
</tbody>
</table>

**Exercise 78 Answer:**

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>−4</td>
<td>0</td>
</tr>
<tr>
<td>−3</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>−180</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

**Exercise 79 Answer:**

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>−8</td>
<td>0</td>
</tr>
<tr>
<td>−6</td>
<td>−36</td>
</tr>
<tr>
<td>−3</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Exercise 80 Answer:**

In Exercises 81–84, sketch a parabola that satisfies the given conditions.

**81.** x-intercepts: −4 and 2; range: y ≥ −3

**82.** Axis of symmetry: x = 6; passes through (4, 15)

**83.** Range: y ≤ 5; passes through (0, 2)

**84.** x-intercept: 6; y-intercept: 1; range: y ≥ −4

**85.** MODELING WITH MATHEMATICS Satellite dishes are shaped like parabolas to optimally receive signals. The cross section of a satellite dish can be modeled by the function shown, where x and y are measured in feet. The x-axis represents the top of the opening of the dish.

**Exercise 85 Answer:**

- **a.** How wide is the satellite dish?
- **b.** How deep is the satellite dish?
- **c.** Write a quadratic function in standard form that models the cross section of a satellite dish that is 6 feet wide and 1.5 feet deep.

**86.** MODELING WITH MATHEMATICS A professional basketball player’s shot is modeled by the function shown, where x and y are measured in feet.

**Exercise 86 Answer:**

- **a.** Does the player make the shot? Explain.
- **b.** The basketball player releases another shot from the point (13, 0) and makes the shot. The shot also passes through the point (10, 1.4). Write a quadratic function in standard form that models the path of the shot.

**87.** y = −x^2 + 5x

**88.** y = x^2 − x − 12

**89.** y = x^3 − 2x^2 − 8x

**90.** y = x^3 − 4x^2 − 11x + 30

**91.** CRITICAL THINKING Write a quadratic function represented by the table, if possible. If not, explain why.

<table>
<thead>
<tr>
<th>x</th>
<th>−5</th>
<th>−3</th>
<th>−1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>0</td>
<td>12</td>
<td>4</td>
<td>0</td>
</tr>
</tbody>
</table>
92. **HOW DO YOU SEE IT?** The graph shows the parabolic arch that supports the roof of a convention center, where \(x\) and \(y\) are measured in feet.

![Graph of the arch]

(a) The arch can be represented by a function of the form \(f(x) = a(x - p)(x - q)\). Estimate the values of \(p\) and \(q\).

(b) Estimate the width and height of the arch. Explain how you can use your height estimate to calculate \(a\).

93. **ANALYZING EQUATIONS** In Exercises 93 and 94, (a) rewrite the quadratic function in intercept form and (b) graph the function using any method. Explain the method you used.

94. \[f(x) = -(x + 1)^2 + 27\]

95. \[g(x) = 2(x - 1)^2 - 2\]

96. **WRITING** Can a quadratic function with exactly one real zero be written in intercept form? Explain.

97. **MAKING AN ARGUMENT** Your friend claims that any quadratic function can be written in standard form and in vertex form. Is your friend correct? Explain.

98. **THOUGHT PROVOKING** Sketch the graph of each function. Explain your procedure.

(a) \[f(x) = (x^2 - 1)(x^2 - 4)\]

(b) \[g(x) = x(x^2 - 1)(x^2 - 4)\]

99. **REASONING** Let \(k\) be a constant. Find the zeros of the function \(f(x) = kx^2 - k^2x - 2k^3\) in terms of \(k\).

**PROBLEM SOLVING** In Exercises 100 and 101, write a system of two quadratic equations whose graphs intersect at the given points. Explain your reasoning.

100. \((-4, 0)\) and \((2, 0)\)

101. \((3, 6)\) and \((7, 6)\)

**Maintaining Mathematical Proficiency** Reviewing what you learned in previous grades and lessons

The scatter plot shows the amounts \(x\) (in grams) of fat and the numbers \(y\) of calories in 12 burgers at a fast-food restaurant. (Section 4.4)

102. How many calories are in the burger that contains 12 grams of fat?

103. How many grams of fat are in the burger that contains 600 calories?

104. What tends to happen to the number of calories as the number of grams of fat increases?

Determine whether the sequence is arithmetic, geometric, or neither. Explain your reasoning. (Section 6.6)

105. 3, 11, 21, 33, 47, . . .

106. \(-2, -6, -18, -54, . . .\)

107. 26, 18, 10, 2, \(-6, . . .\)

108. 4, 5, 9, 14, 23, . . .