10.2 Graphing Cube Root Functions

Essential Question What are some of the characteristics of the graph of a cube root function?

Exploration 1 Graphing Cube Root Functions

Work with a partner.

• Make a table of values for each function. Use positive and negative values of $x$.
• Use the table to sketch the graph of each function.
• Describe the domain of each function.
• Describe the range of each function.

a. $y = \sqrt[3]{x}$

b. $y = \sqrt[3]{x} + 3$

Exploration 2 Writing Cube Root Functions

Work with a partner. Write a cube root function, $y = f(x)$, that has the given values. Then use the function to complete the table.

a. $x$ $f(x)$
   -4 0
   -3
   -2
   -1 $\sqrt[3]{3}$
   0

b. $x$ $f(x)$
   -4 1
   -3
   -2
   -1 $1 + \sqrt[3]{3}$
   0

Communicate Your Answer

3. What are some of the characteristics of the graph of a cube root function?

4. Graph each function. Then compare the graph to the graph of $f(x) = \sqrt[3]{x}$.

   a. $g(x) = \sqrt[3]{x} - 1$
   b. $g(x) = \sqrt[3]{x} - 1$
   c. $g(x) = 2\sqrt[3]{x}$
   d. $g(x) = -2\sqrt[3]{x}$
10.2 Lesson

What You Will Learn

- Graph cube root functions.
- Compare cube root functions using average rates of change.
- Solve real-life problems involving cube root functions.

Graphing Cube Root Functions

Core Vocabulary

cube root function, p. 552

Previous
radical function
index

Core Concept

Cube Root Functions

A cube root function is a radical function with an index of 3. The parent function for the family of cube root functions is \( f(x) = \sqrt[3]{x} \). The domain and range of \( f \) are all real numbers.

The graph of \( f(x) = \sqrt[3]{x} \) increases on the entire domain.

You can transform graphs of cube root functions in the same way you transformed graphs of square root functions.

Example 1: Comparing Graphs of Cube Root Functions

Graph \( h(x) = \sqrt[3]{x} - 4 \). Compare the graph to the graph of \( f(x) = \sqrt[3]{x} \).

Solution

Step 1 Make a table of values.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-8</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h(x) )</td>
<td>-6</td>
<td>-5</td>
<td>-4</td>
<td>-3</td>
<td>-2</td>
</tr>
</tbody>
</table>

Step 2 Plot the ordered pairs.

Step 3 Draw a smooth curve through the points.

The graph of \( h \) is a translation 4 units down of the graph of \( f \).

Monitoring Progress

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Graph the function. Compare the graph to the graph of \( f(x) = \sqrt[3]{x} \).

1. \( h(x) = \sqrt[3]{x} + 3 \)
2. \( m(x) = \sqrt[3]{x} - 5 \)
3. \( g(x) = 4\sqrt[3]{x} \)
**EXAMPLE 2** Comparing Graphs of Cube Root Functions

Graph \( g(x) = -\sqrt[3]{x} + 2 \). Compare the graph to the graph of \( f(x) = \sqrt[3]{x} \).

**SOLUTION**

**Step 1** Make a table of values.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-10</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g(x) )</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>-2</td>
</tr>
</tbody>
</table>

**Step 2** Plot the ordered pairs.

**Step 3** Draw a smooth curve through the points.

The graph of \( g \) is a translation 2 units left and a reflection in the \( x \)-axis of the graph of \( f \).

**EXAMPLE 3** Graphing \( y = a \sqrt[3]{x - h} + k \)

Let \( g(x) = 2\sqrt[3]{x - 3} + 4 \). (a) Describe the transformations from the graph of \( f(x) = \sqrt[3]{x} \) to the graph of \( g \). (b) Graph \( g \).

**SOLUTIION**

a. **Step 1** Translate the graph of \( f \) horizontally 3 units right to get the graph of \( t(x) = \sqrt[3]{x - 3} \).

**Step 2** Stretch the graph of \( t \) vertically by a factor of 2 to get the graph of \( h(x) = 2\sqrt[3]{x - 3} \).

**Step 3** Because \( a > 0 \), there is no reflection.

**Step 4** Translate the graph of \( h \) vertically 4 units up to get the graph of \( g(x) = 2\sqrt[3]{x - 3} + 4 \).

b. **Step 1** Make a table of values.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-5</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g(x) )</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
</tr>
</tbody>
</table>

**Step 2** Plot the ordered pairs.

**Step 3** Draw a smooth curve through the points.

**Monitoring Progress**

Graph the function. Compare the graph to the graph of \( f(x) = \sqrt[3]{x} \).

4. \( g(x) = 0.5\sqrt[3]{x} + 5 \)  
5. \( h(x) = 4\sqrt[3]{x} - 1 \)  
6. \( n(x) = \sqrt[3]{4 - x} \)

7. Let \( g(x) = -\frac{1}{2}\sqrt[3]{x} + 2 - 4 \). Describe the transformations from the graph of \( f(x) = \sqrt[3]{x} \) to the graph of \( g \). Then graph \( g \).
Comparing Average Rates of Change

**EXAMPLE 4** Comparing Cube Root Functions

The graph of cube root function \( m \) is shown. Compare the average rate of change of \( m \) to the average rate of change of \( h(x) = \frac{1}{4} \sqrt[3]{-1 - 4x} \) over the interval \( x = 0 \) to \( x = 8 \).

**SOLUTION**

To calculate the average rates of change, use points whose \( x \)-coordinates are 0 and 8.

Function \( m \): Use the graph to estimate. Use (0, 0) and (8, 8).

\[
\frac{m(8) - m(0)}{8 - 0} \approx \frac{8 - 0}{8} = 1 \quad \text{Average rate of change of } m
\]

Function \( h \): Evaluate \( h \) when \( x = 0 \) and \( x = 8 \).

\[
h(0) = \frac{1}{4} (0) = 0 \quad \text{and} \quad h(8) = \frac{1}{4} (8) = \sqrt[3]{2} \approx 1.3
\]

Use (0, 0) and \((8, \sqrt[3]{2})\).

\[
\frac{h(8) - h(0)}{8 - 0} = \frac{\sqrt[3]{2} - 0}{8} \approx 0.16 \quad \text{Average rate of change of } h
\]

The average rate of change of \( m \) is \( 1 \div \frac{\sqrt[3]{2}}{8} \approx 6.3 \) times greater than the average rate of change of \( h \) over the interval \( x = 0 \) to \( x = 8 \).

**Monitoring Progress**

8. In Example 4, compare the average rates of change over the interval \( x = 2 \) to \( x = 10 \).

Solving Real-Life Problems

**EXAMPLE 5** Real-Life Application

The shoulder height \( h \) (in centimeters) of a male Asian elephant can be modeled by the function \( h = 62.5 \sqrt[3]{t} + 75.8 \), where \( t \) is the age (in years) of the elephant. Use a graphing calculator to graph the function. Estimate the age of an elephant whose shoulder height is 200 centimeters.

**SOLUTION**

Step 1 Enter \( y_1 = 62.5 \sqrt[3]{t} + 75.8 \) and \( y_2 = 200 \) into your calculator and graph the equations. Choose a viewing window that shows the point where the graphs intersect.

Step 2 Use the intersect feature to find the \( x \)-coordinate of the intersection point.

The two graphs intersect at about (8, 200). So, the elephant is about 8 years old.

**Monitoring Progress**

9. **WHAT IF?** Estimate the age of an elephant whose shoulder height is 175 centimeters.
10.2 Exercises

Vocabulary and Core Concept Check
1. COMPLETE THE SENTENCE The ________ of the radical in a cube root function is 3.
2. WRITING Describe the domain and range of the function \( f(x) = \sqrt[3]{x} - 4 + 1 \).

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, match the function with its graph.

3. \( y = \sqrt[3]{x} + 2 \)  
4. \( y = \sqrt[3]{x} - 2 \)  
5. \( y = \sqrt[3]{x} + 2 \)  
6. \( y = \sqrt[3]{x} - 2 \)

A. \[ \text{Graph 1} \]  
B. \[ \text{Graph 2} \]  
C. \[ \text{Graph 3} \]  
D. \[ \text{Graph 4} \]

In Exercises 7–12, graph the function. Compare the graph to the graph of \( f(x) = \sqrt[3]{x} \). (See Example 1.)

7. \( h(x) = \sqrt[3]{x} - 4 \)  
8. \( g(x) = \sqrt[3]{x} + 1 \)  
9. \( m(x) = \sqrt[3]{x} + 5 \)  
10. \( q(x) = \frac{1}{3} \sqrt[3]{x} - 3 \)  
11. \( p(x) = 6\sqrt[3]{x} \)  
12. \( j(x) = \frac{1}{3} \sqrt[3]{x} \)

In Exercises 13–16, compare the graphs. Find the value of \( h, k, \) or \( a \).

13. \( q(x) = \frac{1}{3} \sqrt[3]{x} - h \)  
14. \( g(x) = \frac{1}{3} \sqrt[3]{x} + k \)

15. \[ \text{Graph 5} \]  
16. \[ \text{Graph 6} \]

In Exercises 17–26, graph the function. Compare the graph to the graph of \( f(x) = \sqrt[3]{x} \). (See Example 2.)

17. \( r(x) = -\sqrt[3]{x} - 2 \)  
18. \( h(x) = -\sqrt[3]{x} + 3 \)  
19. \( k(x) = 5\sqrt[3]{x} + 1 \)  
20. \( j(x) = 0.5\sqrt[3]{x} - 4 \)  
21. \( g(x) = 4\sqrt[3]{x} - 3 \)  
22. \( m(x) = 3\sqrt[3]{x} + 7 \)  
23. \( n(x) = \sqrt[3]{-8x} - 1 \)  
24. \( v(x) = \sqrt[3]{5x} + 2 \)  
25. \( q(x) = \sqrt[3]{2(x + 3)} \)  
26. \( p(x) = \sqrt[3]{3(1 - x)} \)

In Exercises 27–32, describe the transformations from the graph of \( f(x) = \sqrt[3]{x} \) to the graph of the given function. Then graph the given function. (See Example 3.)

27. \( g(x) = \sqrt[3]{x} - 4 + 2 \)  
28. \( n(x) = \sqrt[3]{x} + 1 - 3 \)  
29. \( j(x) = -5\sqrt[3]{x} + 3 + 2 \)  
30. \( k(x) = 6\sqrt[3]{x} - 9 - 5 \)  
31. \( v(x) = \frac{1}{3} \sqrt[3]{x} - 1 + 7 \)  
32. \( h(x) = \frac{1}{3} \sqrt[3]{x} + 4 - 3 \)

33. ERROR ANALYSIS Describe and correct the error in graphing the function \( f(x) = \sqrt[3]{x} - 3 \).
34. **ERROR ANALYSIS** Describe and correct the error in graphing the function \( h(x) = \sqrt[3]{x} + 1 \).

35. **COMPARING FUNCTIONS** The graph of cube root function \( q \) is shown. Compare the average rate of change of \( q \) to the average rate of change of \( f(x) = 3\sqrt[3]{x} \) over the interval \( x = 0 \) to \( x = 6 \). (See Example 4.)

36. **COMPARING FUNCTIONS** The graphs of two cube root functions are shown. Compare the average rates of change of the two functions over the interval \( x = -2 \) to \( x = 2 \).

37. **MODELING WITH MATHEMATICS** For a drag race car that weighs 1600 kilograms, the velocity \( v \) (in kilometers per hour) reached by the end of a drag race can be modeled by the function \( v = 23.8\sqrt{p} \), where \( p \) is the car’s power (in horsepower). Use a graphing calculator to graph the function. Estimate the power of a 1600-kilogram car that reaches a velocity of 220 kilometers per hour. (See Example 5.)

38. **MODELING WITH MATHEMATICS** The radius \( r \) of a sphere is given by the function \( r = \frac{\sqrt[3]{3V}}{4\pi} \), where \( V \) is the volume of the sphere. Use a graphing calculator to graph the function. Estimate the volume of a spherical beach ball with a radius of 13 inches.

39. **MAKING AN ARGUMENT** Your friend says that all cube root functions are odd functions. Is your friend correct? Explain.

40. **HOW DO YOU SEE IT?** The graph represents the cube root function \( f(x) = \sqrt[3]{x} \).

   a. On what interval is \( f \) negative? positive?
   b. On what interval, if any, is \( f \) decreasing? increasing?
   c. Does \( f \) have a maximum or minimum value? Explain.
   d. Find the average rate of change of \( f \) over the interval \( x = -1 \) to \( x = 1 \).

41. **PROBLEM SOLVING** Write a cube root function that passes through the point \((3, 4)\) and has an average rate of change of \(-1\) over the interval \(x = -5\) to \(x = 2\).

42. **THOUGHT PROVOKING** Write the cube root function represented by the graph. Use a graphing calculator to check your answer.

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**Maintaining Mathematical Proficiency**

Reviewing what you learned in previous grades and lessons

Factor the polynomial. (Section 7.6)

43. \( 3x^2 + 12x - 36 \)  
   44. \( 2x^2 - 11x + 9 \)  
   45. \( 4x^2 + 7x - 15 \)

Solve the equation using square roots. (Section 9.3)

46. \( x^2 - 36 = 0 \)  
   47. \( 5x^2 + 20 = 0 \)  
   48. \( (x + 4)^2 = 81 \)  
   49. \( 25(x - 2)^2 = 9 \)