5.1 **nth Roots and Rational Exponents**

**Essential Question**  How can you use a rational exponent to represent a power involving a radical?

Previously, you learned that the *n*th root of *a* can be represented as

\[
\sqrt[n]{a} = a^{1/n}
\]

**Definition of rational exponent**

for any real number *a* and integer *n* greater than 1.

**CONSTRUCTING VIABLE ARGUMENTS**

To be proficient in math, you need to understand and use stated definitions and previously established results.

**EXPLORATION 1**  Exploring the Definition of a Rational Exponent

**Work with a partner.** Use a calculator to show that each statement is true.

a. \(\sqrt{9} = 9^{1/2}\)

b. \(\sqrt{2} = 2^{1/2}\)

c. \(\sqrt{8} = 8^{1/3}\)

d. \(\sqrt[3]{3} = 3^{1/3}\)

e. \(\sqrt[4]{16} = 16^{1/4}\)

f. \(\sqrt[4]{12} = 12^{1/4}\)

**EXPLORATION 2**  Writing Expressions in Rational Exponent Form

**Work with a partner.** Use the definition of a rational exponent and the properties of exponents to write each expression as a base with a single rational exponent. Then use a calculator to evaluate each expression. Round your answer to two decimal places.

**Sample**

\[
\left(\sqrt[3]{4}\right)^2 = (4^{1/3})^2 = 4^{2/3} \approx 2.52
\]

a. \((\sqrt{5})^3\)

b. \((\sqrt[3]{4})^2\)

c. \((\sqrt{9})^2\)

d. \((\sqrt{10})^4\)

e. \((\sqrt[4]{15})^3\)

f. \((\sqrt[4]{27})^4\)

**EXPLORATION 3**  Writing Expressions in Radical Form

**Work with a partner.** Use the properties of exponents and the definition of a rational exponent to write each expression as a radical raised to an exponent. Then use a calculator to evaluate each expression. Round your answer to two decimal places.

**Sample**

\[
5^{2/3} = (5^{1/3})^2 = (\sqrt[3]{5})^2 \approx 2.92
\]

a. \(8^{2/3}\)

b. \(6^{3/2}\)

c. \(12^{3/4}\)

d. \(10^{3/2}\)

e. \(16^{3/2}\)

f. \(20^{6/3}\)

**Communicate Your Answer**

4. How can you use a rational exponent to represent a power involving a radical?

5. Evaluate each expression *without* using a calculator. Explain your reasoning.

a. \(4^{3/2}\)

b. \(32^{4/5}\)

c. \(625^{3/4}\)

d. \(49^{3/2}\)

e. \(125^{4/3}\)

f. \(100^{6/3}\)
5.1 Lesson

What You Will Learn

- Find $n$th roots of numbers.
- Evaluate expressions with rational exponents.
- Solve equations using $n$th roots.

$n$th Roots

You can extend the concept of a square root to other types of roots. For example, 2 is a cube root of 8 because $2^3 = 8$. In general, for an integer $n$ greater than 1, if $b^n = a$, then $b$ is an $n$th root of $a$. An $n$th root of $a$ is written as $\sqrt[n]{a}$, where $n$ is the index of the radical.

You can also write an $n$th root of $a$ as a power of $a$. If you assume the Power of a Power Property applies to rational exponents, then the following is true.

- $(a^{1/2})^2 = a^{1/2 \cdot 2} = a^{1} = a$
- $(a^{1/3})^3 = a^{1/3 \cdot 3} = a^{1} = a$
- $(a^{1/4})^4 = a^{1/4 \cdot 4} = a^{1} = a$

Because $a^{1/2}$ is a number whose square is $a$, you can write $\sqrt{a} = a^{1/2}$. Similarly, $\sqrt[3]{a} = a^{1/3}$ and $\sqrt[4]{a} = a^{1/4}$. In general, $\sqrt[n]{a} = a^{1/n}$ for any integer $n$ greater than 1.

Understanding Mathematical Terms

When $n$ is even and $a > 0$, there are two real roots. The positive root is called the principal root.

Core Concept

Real $n$th Roots of $a$

Let $n$ be an integer ($n > 1$) and let $a$ be a real number.

- $n$ is an even integer.
  - $a < 0$ No real $n$th roots
  - $a = 0$ One real $n$th root: $\sqrt[n]{0} = 0$
  - $a > 0$ Two real $n$th roots: $\pm \sqrt[n]{a} = \pm a^{1/n}$
- $n$ is an odd integer.
  - $a < 0$ One real $n$th root: $\sqrt[n]{a} = a^{1/n}$
  - $a = 0$ One real $n$th root: $\sqrt[0]{0} = 0$
  - $a > 0$ One real $n$th root: $\sqrt[n]{a} = a^{1/n}$

Example 1 Finding $n$th Roots

Find the indicated real $n$th root(s) of $a$.

a. $n = 3, a = -216$ 

b. $n = 4, a = 81$

Solution

a. Because $n = 3$ is odd and $a = -216 < 0$, $-216$ has one real cube root.
   Because $(-6)^3 = -216$, you can write $\sqrt[3]{-216} = -6$ or $(-216)^{1/3} = -6$.

b. Because $n = 4$ is even and $a = 81 > 0$, $81$ has two real fourth roots.
   Because $3^4 = 81$ and $(-3)^4 = 81$, you can write $\pm \sqrt[4]{81} = \pm 3$ or $\pm 81^{1/4} = \pm 3$.

Monitoring Progress

Find the indicated real $n$th root(s) of $a$.

1. $n = 4, a = 16$
2. $n = 2, a = -49$
3. $n = 3, a = -125$
4. $n = 5, a = 243$
Rational Exponents

A rational exponent does not have to be of the form \( \frac{1}{n} \). Other rational numbers, such as \( \frac{3}{2} \) and \( -\frac{1}{2} \), can also be used as exponents. Two properties of rational exponents are shown below.

**Core Concept**

**Rational Exponents**

Let \( a^{\frac{1}{n}} \) be an \( n \)th root of \( a \), and let \( m \) be a positive integer.

\[
\begin{align*}
    a^m &= (a^{\frac{1}{n}})^m = (\sqrt[n]{a})^m \\
    a^{-m} &= \frac{1}{a^m} = \frac{1}{(a^{\frac{1}{n}})^m} = \frac{1}{(\sqrt[n]{a})^m} \quad a \neq 0
\end{align*}
\]

**EXAMPLE 2** Evaluating Expressions with Rational Exponents

Evaluate each expression.

a. \( 16^{\frac{3}{2}} \)  

b. \( 32^{-\frac{3}{5}} \)

**SOLUTION**

Rational Exponent Form | Radical Form
--- | ---

a. \( 16^{\frac{3}{2}} = (16^{\frac{1}{2}})^3 = 4^3 = 64 \)  
\( 16^{\frac{3}{2}} = (\sqrt{16})^3 = 4^3 = 64 \)

b. \( 32^{-\frac{3}{5}} = \frac{1}{32^{\frac{3}{5}}} = \frac{1}{(32^{\frac{1}{5}})^3} = \frac{1}{2^3} = \frac{1}{8} \)  
\( 32^{-\frac{3}{5}} = \frac{1}{(\sqrt[5]{32})^3} = \frac{1}{2^3} = \frac{1}{8} \)

When using a calculator to approximate an \( n \)th root, you may want to rewrite the \( n \)th root in rational exponent form.

**EXAMPLE 3** Approximating Expressions with Rational Exponents

Evaluate each expression using a calculator. Round your answer to two decimal places.

a. \( 9^{\frac{1}{5}} \)  

b. \( 12^{\frac{3}{8}} \)

c. Before evaluating \( \left(\sqrt[3]{7}\right)^3 \), rewrite the expression in rational exponent form.

\( \left(\sqrt[3]{7}\right)^3 = 7^{\frac{3}{4}} = 4.30 \)

**Monitoring Progress**

Evaluate the expression without using a calculator.

5. \( 4^{\frac{5}{2}} \)  
6. \( 9^{-\frac{1}{2}} \)  
7. \( 81^{\frac{3}{4}} \)  
8. \( 1^{\frac{7}{8}} \)

Evaluate the expression using a calculator. Round your answer to two decimal places when appropriate.

9. \( 6^{\frac{2}{5}} \)  
10. \( 64^{-\frac{2}{3}} \)  
11. \( \left(\sqrt[5]{16}\right)^5 \)  
12. \( \left(\sqrt[3]{-30}\right)^2 \)
Solving Equations Using \( n \)th Roots

To solve an equation of the form \( u^n = d \), where \( u \) is an algebraic expression, take the \( n \)th root of each side.

**EXAMPLE 4** Solving Equations Using \( n \)th Roots

Find the real solution(s) of (a) \( 4x^5 = 128 \) and (b) \( (x - 3)^4 = 21 \).

**SOLUTION**

**a.** \( 4x^5 = 128 \)

\[
4x^5 = 128
\]

\[
x^5 = 32
\]

\[
x = \sqrt[5]{32}
\]

\[
x = 2
\]

The solution is \( x = 2 \).

**b.** \( (x - 3)^4 = 21 \)

\[
(x - 3)^4 = 21
\]

\[
x - 3 = \pm \sqrt[4]{21}
\]

\[
x = 3 \pm \sqrt[4]{21}
\]

\[
x \approx 3 + \sqrt[4]{21} \quad \text{or} \quad x \approx 3 - \sqrt[4]{21}
\]

The solutions are \( x \approx 5.14 \) and \( x \approx 0.86 \).

**Real-Life Application**

A hospital purchases an ultrasound machine for $50,000. The hospital expects the useful life of the machine to be 10 years, at which time its value will have depreciated to $8000. The hospital uses the declining balances method for depreciation, so the annual depreciation rate \( r \) (in decimal form) is given by the formula

\[
r = 1 - \left( \frac{S}{C} \right)^{1/n}.
\]

In the formula, \( n \) is the useful life of the item (in years), \( S \) is the salvage value (in dollars), and \( C \) is the original cost (in dollars). What annual depreciation rate did the hospital use?

**SOLUTION**

The useful life is 10 years, so \( n = 10 \). The machine depreciates to $8000, so \( S = 8000 \). The original cost is $50,000, so \( C = 50,000 \). So, the annual depreciation rate is

\[
r = 1 - \left( \frac{8000}{50,000} \right)^{1/10} = 1 - \left( \frac{4}{25} \right)^{1/10} \approx 0.167.
\]

The annual depreciation rate is about 0.167, or 16.7%.

**Monitoring Progress**

Find the real solution(s) of the equation. Round your answer to two decimal places when appropriate.

13. \( 8x^3 = 64 \)  
14. \( \frac{1}{2}x^5 = 512 \)  
15. \( (x + 5)^4 = 16 \)  
16. \( (x - 2)^3 = -14 \)

17. **WHAT IF?** In Example 5, what is the annual depreciation rate when the salvage value is $6000?
5.1 Exercises

Vocabulary and Core Concept Check

1. **VOCABULARY** Rewrite the expression $a^{-\frac{s}{t}}$ in radical form. Then state the index of the radical.

2. **COMPLETE THE SENTENCE** For an integer $n$ greater than 1, if $b^n = a$, then $b$ is a(n) ___________ of $a$.

3. **WRITING** Explain how to use the sign of $a$ to determine the number of real fourth roots of $a$ and the number of real fifth roots of $a$.

4. **WHICH ONE DOESN'T BELONG?** Which expression does not belong with the other three? Explain your reasoning.

\[
\begin{align*}
(a^{1/n})^m & \quad (\sqrt[n]{a})^m \\
(a^{n})^{-n} & \quad a^{m/n}
\end{align*}
\]

Monitoring Progress and Modeling with Mathematics

In Exercises 5–10, find the indicated real $n$th root(s) of $a$. (See Example 1.)

5. $n = 3, \ a = 8$
6. $n = 5, \ a = -1$
7. $n = 2, \ a = 0$
8. $n = 4, \ a = 256$
9. $n = 5, \ a = -32$
10. $n = 6, \ a = -729$

In Exercises 11–18, evaluate the expression without using a calculator. (See Example 2.)

11. $64^{1/6}$
12. $8^{1/3}$
13. $25^{3/2}$
14. $81^{3/4}$
15. $(-243)^{1/5}$
16. $(-64)^{4/3}$
17. $8^{-2/3}$
18. $16^{-7/4}$

**ERROR ANALYSIS** In Exercises 19 and 20, describe and correct the error in evaluating the expression.

19. \[27^{2/3} = (27^{1/3})^2\]

\[= 9^2\]
\[= 81\]

20. \[256^{4/3} = (\sqrt[3]{256})^3\]

\[= 4^3\]
\[= 64\]

**USING STRUCTURE** In Exercises 21–24, match the equivalent expressions. Explain your reasoning.

21. $\left(\sqrt[4]{5}\right)^4$
   A. $5^{-1/4}$
22. $\left(\sqrt[3]{5}\right)^3$
   B. $5^{4/3}$
23. $\frac{1}{\sqrt[5]{5}}$
   C. $-5^{1/4}$
24. $-\sqrt[5]{5}$
   D. $5^{3/4}$

In Exercises 25–32, evaluate the expression using a calculator. Round your answer to two decimal places when appropriate. (See Example 3.)

25. $\sqrt[4]{32,768}$
26. $\sqrt[4]{1695}$
27. $25^{-1/3}$
28. $85^{1/6}$
29. $20,736^{4/5}$
30. $86^{-5/6}$
31. $\left(\sqrt[3]{187}\right)^3$
32. $\left(\sqrt[3]{-8}\right)^8$

**MATHEMATICAL CONNECTIONS** In Exercises 33 and 34, find the radius of the figure with the given volume.

33. $V = 216 \text{ ft}^3$
34. $V = 1332 \text{ cm}^3$

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In Exercises 35–44, find the real solution(s) of the equation. Round your answer to two decimal places when appropriate. (See Example 4.)

35. \(x^3 = 125\)  
36. \(5x^3 = 1080\)

37. \((x + 10)^5 = 70\)  
38. \((x - 5)^4 = 256\)

39. \(x^5 = -48\)  
40. \(7x^4 = 56\)

41. \(x^6 + 36 = 100\)  
42. \(x^3 + 40 = 25\)

43. \(\frac{1}{3}x^4 = 27\)  
44. \(\frac{1}{2}x^3 = -36\)

45. **MODELING WITH MATHEMATICS** When the average price of an item increases from \(p_1\) to \(p_2\) over a period of \(n\) years, the annual rate of inflation \(r\) (in decimal form) is given by \(r = \left(\frac{p_2}{p_1}\right)^{\frac{1}{n}} - 1\). Find the rate of inflation for each item in the table. (See Example 5.)

<table>
<thead>
<tr>
<th>Item</th>
<th>Price in 1913</th>
<th>Price in 2013</th>
</tr>
</thead>
<tbody>
<tr>
<td>Potatoes (lb)</td>
<td>$0.016</td>
<td>$0.627</td>
</tr>
<tr>
<td>Ham (lb)</td>
<td>$0.251</td>
<td>$2.693</td>
</tr>
<tr>
<td>Eggs (dozen)</td>
<td>$0.373</td>
<td>$1.933</td>
</tr>
</tbody>
</table>

46. **HOW DO YOU SEE IT?** The graph of \(y = x^n\) is shown in red. What can you conclude about the value of \(n\)? Determine the number of real \(n\)th roots of \(a\). Explain your reasoning.

47. **NUMBER SENSE** Between which two consecutive integers does \(\sqrt[3]{125}\) lie? Explain your reasoning.

48. **THOUGHT PROVOKING** In 1619, Johannes Kepler published his third law, which can be given by \(d^3 = t^2\), where \(d\) is the mean distance (in astronomical units) of a planet from the Sun and \(t\) is the time (in years) it takes the planet to orbit the Sun. It takes Mars 1.88 years to orbit the Sun. Graph a possible location of Mars. Justify your answer. (The diagram shows the Sun at the origin of the \(xy\)-plane and a possible location of Earth.)

49. **PROBLEM SOLVING** A *weir* is a dam that is built across a river to regulate the flow of water. The flow rate \(Q\) (in cubic feet per second) can be calculated using the formula \(Q = 3.367\ell h^{3/2}\), where \(\ell\) is the length (in feet) of the bottom of the spillway and \(h\) is the depth (in feet) of the water on the spillway. Determine the flow rate of a weir with a spillway that is 20 feet long and has a water depth of 5 feet.

50. **REPEATED REASONING** The mass of the particles that a river can transport is proportional to the sixth power of the speed of the river. A certain river normally flows at a speed of 1 meter per second. What must its speed be in order to transport particles that are twice as massive as usual? 10 times as massive? 100 times as massive?

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**Maintaining Mathematical Proficiency**

Reviewing what you learned in previous grades and lessons

Simplify the expression. Write your answer using only positive exponents. *(Skills Review Handbook)*

51. \(5 \cdot 5^4\)  
52. \(\frac{4^2}{4^3}\)  
53. \((z^2)^{-3}\)  
54. \(\left(\frac{3x}{2}\right)^4\)

Write the number in standard form. *(Skills Review Handbook)*

55. \(5 \times 10^3\)  
56. \(4 \times 10^{-2}\)  
57. \(8.2 \times 10^{-1}\)  
58. \(6.93 \times 10^6\)