Analytic Trigonometry, (+) Limits and Continuity

Content Area:	**Math**
Course(s):	
Time Period:	**MP4**
Length:	**45**
Status:	**Published**

Unit Overview

Unit Summary	Unit Rationale
In this unit, students will investigate trigonometry and limits, building on prior mathematical knowledge. They will explore the properties and applications of sine, cosine, and tangent functions, focusing on their relationships to the unit circle and right triangles. Through graphing and simplifying trigonometric identities, students will deepen their understanding of these functions. The unit will also cover limits, if time allows, where students will learn to evaluate them numerically and graphically, emphasizing continuity and asymptotic behavior. Real-world applications will illustrate the relevance of limits in various fields. Throughout the unit, students will enhance their problem-solving skills and critical thinking, preparing them for advanced concepts in calculus. By the end, they will appreciate the interconnectedness of mathematics and its application in understanding the world.	The rationale for this unit on trigonometry and limits is grounded in the necessity of equipping students with essential mathematical skills that are foundational for higher-level mathematics and real-world applications. Understanding trigonometric functions and their properties is crucial, as they are widely used in various fields, including physics, engineering, and computer science. This unit not only reinforces previously learned concepts but also encourages students to make connections between trigonometry and other areas of mathematics, such as geometry and algebra. In addition, the exploration of limits serves as a gateway to calculus, introducing students to the concept of approaching values and the behavior of functions. By engaging with these ideas, students will develop critical thinking and problem-solving skills that are vital for academic success and informed decision-making in everyday life. Ultimately, this unit aims to foster a deep understanding of the interconnectedness of mathematical concepts while preparing students for future challenges in mathematics and related disciplines.

NJSLS

MA.9-12.1.1.CHA-1.A.1	Calculus uses limits to understand and model dynamic change.
MA.9-12.1.1.CHA-1.A.2	Because an average rate of change divides the change in one variable by the change in another, the average rate of change is undefined at a point where the change in the independent variable would be zero.

MA.9-12.1.1.CHA-1.A.3	The limit concept allows us to define instantaneous rate of change in terms of average rates of change.
MA.9-12.1.2.LIM-1.A.1	Given a function f, the limit of $f(x)$ as x approaches c is a real number R if $f(x)$ can be made arbitrarily close to R by taking x sufficiently close to c (but not equal to c). If the limit exists and is a real number, then the common notation is lim [$x \rightarrow c$] $f(x) = R$.
MA.9-12.1.2.LIM-1.B.1	A limit can be expressed in multiple ways, including graphically, numerically, and analytically.
MA.9-12.1.3.LIM-1.C.1	The concept of a limit includes one sided limits.
MA.9-12.1.3.LIM-1.C.2	Graphical information about a function can be used to estimate limits.
MA.9-12.1.3.LIM-1.C.3	Because of issues of scale, graphical representations of functions may miss important function behavior.
MA.9-12.1.3.LIM-1.C.4	A limit might not exist for some functions at particular values of x. Some ways that the limit might not exist are if the function is unbounded, if the function is oscillating near this value, or if the limit from the left does not equal the limit from the right.
MA.9-12.1.4.LIM-1.C.5	Numerical information can be used to estimate limits.
MA.9-12.1.5.LIM-1.D.1	One-sided limits can be determined analytically or graphically.
MA.9-12.1.5.LIM-1.D.2	Limits of sums, differences, products, quotients, and composite functions can be found using limit theorems.
MA.9-12.1.6.LIM-1	Reasoning with definitions, theorems, and properties can be used to justify claims about limits.
MA.9-12.1.6.LIM-1.E.1	It may be necessary or helpful to rearrange expressions into equivalent forms before evaluating limits.
MA.9-12.1.10.LIM-2.A.1	Types of discontinuities include removable discontinuities, jump discontinuities, and discontinuities due to vertical asymptotes.
MA.9-12.1.11.LIM-2.A.2	A function f is continuous at $x = c$ provided that $f(c)$ exists, lim [$x \rightarrow c$] $f(x)$ exists, and lim [$x \rightarrow c$] $f(x) = f(c)$.
MA.9-12.1.12.LIM-2.B.1	A function is continuous on an interval if the function is continuous at each point in the interval.
MA.9-12.1.12.LIM-2.B.2	Polynomial, rational, power, exponential, logarithmic, and trigonometric functions are continuous on all points in their domains.
MA.9-12.1.13.LIM-2.C.1	If the limit of a function exists at a discontinuity in its graph, then it is possible to remove the discontinuity by defining or redefining the value of the function at that point, so it equals the value of the limit of the function as x approaches that point.
MA.9-12.1.14.LIM-2.D.1	The concept of a limit can be extended to include infinite limits.
MATH.9-12.F.TF.B.5	Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.
MA.9-12.1.14.LIM-2.D.2	Asymptotic and unbounded behavior of functions can be described and explained using limits.
MATH.9-12.F.TF.B.6	Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.
MATH.9-12.F.TF.B.7	Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context.
MATH.9-12.F.TF.C.8	Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the angle.
MATH.9-12.F.TF.C.9	Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.
MA.9-12.1.15.LIM-2.D.3	The concept of a limit can be extended to include limits at infinity.

MA.9-12.1.15.LIM-2.D.4	Limits at infinity describe end behavior.
MA.9-12.1.15.LIM-2.D.5	Relative magnitudes of functions and their rates of change can be compared using limits.

Standards for Mathematical Practice

MATH.K-12.1	Make sense of problems and persevere in solving them
MATH.K-12.2	Reason abstractly and quantitatively
MATH.K-12.3	Construct viable arguments and critique the reasoning of others
MATH.K-12.4	Model with mathematics
MATH.K-12.5	Use appropriate tools strategically
MATH.K-12.6	Attend to precision
MATH.K-12.7	Look for and make use of structure
MATH.K-12.8	Look for and express regularity in repeated reasoning

Unit Focus

Enduring Understandings	Essential Questions
• Trigonometric functions model cyclical phenomena and relationships in real-world scenarios, such as wave motion and oscillations. • The periodic nature of trigonometric functions allows for the prediction of behavior over intervals, which is essential for applications in physics and engineering. • The ability to interpret and analyze the graphs of trigonometric functions enhances understanding of their properties, including amplitude, frequency, and phase shifts. • Solving trigonometric equations can be approached both algebraically and graphically, providing multiple methods for finding solutions. • Trigonometric identities illustrate the interconnectedness of trigonometric functions, allowing for simplification and transformation of expressions. • Trigonometric functions have diverse applications in various fields, including engineering, physics, and computer graphics,	• How do trigonometric functions model real-world phenomena, and what are some examples of their applications? • What characteristics of trigonometric functions, such as amplitude and period, affect their graphs and real-world interpretations? • In what ways can graphical and algebraic methods be used interchangeably to solve trigonometric equations, and what are the advantages of each method? • How do trigonometric identities reveal relationships among different functions, and why are these identities important in solving problems? • What is the significance of angle measurement in both degrees and radians for solving trigonometric problems? • How can sinusoidal functions be used to effectively model and solve problems involving simple harmonic motion? • In what ways do the properties of trigonometric functions facilitate the

highlighting their practical importance. • Understanding angle measurement in both degrees and radians is crucial for applying trigonometric concepts accurately. • Sinusoidal functions can effectively model simple harmonic motion, providing tools for solving real-world problems involving oscillations. Additional: • *The concept of limits is foundational to calculus, serving as the basis for understanding continuity, derivatives, and integrals.* • *Limits help describe the behavior of functions as they approach specific points or infinity, providing insight into their properties.* • *A function is continuous if it can be drawn without lifting a pencil; limits are used to determine whether a function meets this criterion at a given point.* • *One-sided limits provide important information about the behavior of functions at points where they may not be defined or where discontinuities occur.* • *Understanding infinite limits is essential for identifying vertical and horizontal asymptotes, which describe the behavior of functions at extremes.* • *The Intermediate Value Theorem illustrates that continuous functions take on every value between their endpoints, ensuring the existence of roots within intervals.* • *Limits can be evaluated through graphical and numerical methods, allowing for multiple pathways to understanding function behavior.* • *Limits are used to define instantaneous rates of change, connecting the concept to real-world applications in motion and growth.*	understanding of periodic behavior in various contexts? • How do we determine the transformations of trigonometric functions based on changes in their parameters, such as shifts and stretches? Additional: • *How do limits serve as a foundation for understanding key concepts in calculus, such as continuity and derivatives?* • *What do limits reveal about the behavior of functions as they approach specific values or infinity?* • *How can we determine whether a function is continuous at a point using limits, and why is continuity important in mathematical analysis?* • *What are one-sided limits, and how do they provide insight into the behavior of functions at points of discontinuity?* • *In what ways do infinite limits relate to the identification of asymptotes, and how do these concepts help describe the behavior of functions?* • *How does the Intermediate Value Theorem demonstrate the relationship between continuity and the existence of roots within an interval?* • *What methods can be employed to evaluate limits, and how do graphical, numerical, and analytical approaches complement each other?* • *How do limits connect to the concept of rate of change, and why is this connection significant in real-world applications?*

Instructional Focus

Learning Targets
• Solve trigonometric equations graphically. • State the complete solution of a trigonometric equation. • Solve trigonometric equations algebraically. • Write a sinusoidal function whose graph resembles a given graph. • Write a sinusoidal function to represent a given simple harmonic motion, and use the function to solve problems. • Identify possible identities by using graphs. • Apply strategies to prove identities. • Use the addition and subtraction identities for sine, cosine, and tangent functions. • Use the cofunction identities. • Apply indities for multiple angles: double-angle, half-angle, product-to-sum, sum-to-product. • Use identities to solve trigonometric equations. • Interpret rate of change at an instant in terms of average rates of change. • *(+) Evaluate a limit, if it exists by graphical and numerical methods.* • *(+) Find one-sided limits and general limits.* • *(+) Use properties of limits to evaluate graphically, numerically, and algebraically.* • *(+) Evaluate limits analytically, including substitution, cancellation, rationalization, and special trig rules.* • *(+) Evaluate limits of trigonometric functions.* • *(+) Apply the definition of continuity to determine whether a function is continuous at a point.* • *(+) Compare and contrast infinite limits and limits at infinity.* • *(+) Use properties of infinite limits to find asymptotes and describe function behavior.*

- *(+) Use relative rates of growth of power, logarithmic, and exponential functions to analyze limits.*
- *(+) Understand and apply the intermediate value theorem.*

Prerequisite Skills

- Knowledge of angles, triangles, and the properties of geometric shapes is essential for grasping trigonometric concepts.
- Students should be able to recognize and interpret different types of functions, including linear, quadratic, and polynomial functions.
- An understanding of the unit circle, including the coordinates of key angles, is crucial for working with trigonometric functions.
- Students should be able to apply the Pythagorean theorem and use sine, cosine, and tangent ratios to solve problems involving right triangles.
- Proficiency in algebra, including simplifying expressions, factoring, and solving equations, is necessary for working with trigonometric identities and equations.
- Experience in graphing functions and interpreting their characteristics (such as intercepts and asymptotes) is important for understanding the behavior of trigonometric graphs.
- Familiarity with square roots and the ability to manipulate radical expressions will aid in solving trigonometric equations.
- Students should be aware of how transformations (shifts, stretches, and reflections) affect the graphs of functions.
- Students should be able to identify and analyze various types of functions and their behaviors as they approach specific points.
- Familiarity with the concept of continuity and the ability to recognize continuous versus discontinuous functions is essential.
- Familiarity with Algebraic Concepts: Skills in manipulating algebraic expressions, including factoring and simplifying, are necessary for evaluating limits.
- Students should be able to interpret graphs to understand limits visually and identify asymptotic behavior.
- Students should have a conceptual understanding of infinity and how it relates to limits, particularly in the context of approaching very large or very small values.
- Students should possess strong analytical and problem-solving skills to tackle complex limit problems, utilizing multiple approaches as needed.

Common Misconceptions

- Students often struggle to switch between degrees and radians, leading to incorrect calculations and interpretations of angles.

- Some students may not grasp the significance of the unit circle, leading to confusion about the values of sine, cosine, and tangent at various angles.
- Students might mistakenly assume that all trigonometric functions behave similarly without recognizing their unique characteristics.
- Students may misuse trigonometric identities, either by not applying them correctly or by failing to recognize when they are applicable.
- Some students may apply trigonometric ratios only to right triangles and struggle with problems involving non-right triangles, neglecting the Law of Sines and the Law of Cosines.
- Students often misinterpret the effects of transformations on the graphs of trigonometric functions, leading to incorrect representations of amplitude, phase shift, and period.
- Students may overlook the domain and range of trigonometric functions, not recognizing that certain values are undefined or restricted.
- Students might struggle with simplifying trigonometric expressions, leading to incorrect conclusions about the relationships between different functions.
- *(+) Students often think a limit is the actual value of a function at a point rather than the value that the function approaches as it gets closer to that point.*
- *(+) Some students may not fully understand the difference between left-hand and right-hand limits, leading to incorrect conclusions about continuity and behavior at critical points.*
- *(+) Students might believe that limits exist for every function at every point, overlooking situations where limits are undefined or do not exist.*
- *(+) Students may fail to recognize how limits relate to vertical and horizontal asymptotes, leading to misunderstandings about the behavior of functions at infinity.*
- *(+) Some students may misunderstand the relationship between limits and continuity, mistakenly thinking that a function can be continuous even if the limits do not match the function's value at a point.*
- *(+) Students may incorrectly apply algebraic operations (such as addition or multiplication) to limits without understanding the underlying rules, leading to miscalculations.*
- *(+) Students might misinterpret graphs when evaluating limits, overlooking asymptotic behavior or discontinuities that affect the limit's value.*
- *(+) Some students may misunderstand the Intermediate Value Theorem, believing that it applies only to linear functions rather than any continuous function.*

Spiraling For Mastery

<table>
<tr><th>Current Unit Content/Skills</th><th>Spiral Focus</th><th>Activity</th></tr>
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Identities and Proofs
Addition and subtraction identities
Cofunction identities
Double-angle identities
Power reducing identities
Half-angle identities
Product-to-Sum and Sum-to-Product identities
Using identities to solve trigonometric equations

Additional:

Instantaneous rate of change
Evaluate a limit using a graph and a table
Find one-sided limits
Properties of limits
Evaluate limits algebraically.
Use substitution, cancellation, rationalization and special trig rules
Define continuity
Discuss infinite limits and limits at infinity
Find asymptotes and describe function behavior

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Students will revisit the understanding of functions, their properties, and how to analyze them. This includes recognizing different types of functions (linear, quadratic, exponential) and understanding their behaviors.
The focus on function transformations (shifts, stretches, reflections) will be essential in both trigonometric graphs and limits.
Skills in simplifying and manipulating algebraic expressions will continue to be critical, particularly when working with trigonometric identities and evaluating limits.
Factoring techniques and solving equations will be revisited as students encounter more complex problems.
The relationship between angles, triangles, and trigonometric functions draws upon prior geometric knowledge, especially regarding right triangles and the Pythagorean theorem.
Understanding the unit circle relies on previous geometry concepts, and students will apply this

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iXL Diagnostic Assessment
iXL Problems
Delta Math

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with limits • *Intermediate Value Theorem*	knowledge to solve trigonometric problems. • Students will apply their graphing skills from earlier units when analyzing and creating graphs of trigonometric functions and understanding their characteristics. • The concept of limits will also involve graphical interpretation, requiring students to visualize and analyze function behavior near specific points. • The use of mathematical concepts to model real-world situations will again be emphasized, particularly in trigonometry through applications like wave motion, oscillations, and periodic phenomena. • Limits will also be connected to real-life scenarios, such as calculating rates of change in various contexts.	

Assessment

Formative Assessment	**Summative Assessment**
• Homework • Lesson Checks • Quizzes • Exit Tickets • Lesson Reflections	Part 1 Assessment - Solving Trigonometric Equations Part 2 Assessment - Trigonometric Identities and Proof Benchmark 4 (Linkit) Honors/AP Benchmark 4 - with Limits (Linkit)

• Performance Tasks	

Resources

Key Resources	Supplemental Resources
• *Pre-Calculus: A Graphing Approach*, Holt, Rinehart and Winston 2007, Chapters 8 and 9 • (+) *Pre-Calculus: A Graphing Approach,* Holt, Rinehart and Winston 2007, Chapter 14 ○ *Intro to Calculus - This section is optional depending on how the year goes with your class. The goal of this topic is to prepare students who plan to take Calculus senior year or after graduation.*	iXL Delta Math Desmos Activity Builder Desmos Graphing Calculator Explorations Khan Academy Teacher made Worksheets APSI Resources for AP Precalculus APSI Resources for AP Calculus

Career Readiness, Life Literacies, and Key Skills

CRP.K-12.CRP2	Apply appropriate academic and technical skills.
CRP.K-12.CRP4	Communicate clearly and effectively and with reason.
CRP.K-12.CRP6	Demonstrate creativity and innovation.
CRP.K-12.CRP7	Employ valid and reliable research strategies.
CRP.K-12.CRP8	Utilize critical thinking to make sense of problems and persevere in solving them.
CRP.K-12.CRP11	Use technology to enhance productivity.
CRP.K-12.CRP12	Work productively in teams while using cultural global competence.

Interdisciplinary Connections

ELA.RL.CR.11–12.1	Accurately cite strong and thorough textual evidence and make relevant connections to strongly support a comprehensive analysis of multiple aspects of what a literary text says explicitly and inferentially, as well as interpretations of the text; this may include determining where the text leaves matters uncertain.
ELA.W.AW.11–12.1	Write arguments to support claims in an analysis of substantive topics or texts, using valid reasoning and relevant and sufficient evidence.
SOC.6.1.16	Contemporary United States: Interconnected Global Society (1970–Today)

CS.9-12.AP	Algorithms & Programming
VPA.1.3.12	All students will synthesize those skills, media, methods, and technologies appropriate to creating, performing, and/or presenting works of art in dance, music, theatre, and visual art.
9-12.HS-ESS2	Earth's Systems
9-12.HS-PS2	Motion and Stability: Forces and Interactions