Conceptual and Analytic Differentiation

Content Area:	**Math**
Course(s):	
Time Period:	**MP2**
Length:	**45**
Status:	**Published**

Unit Overview

Unit Summary	Unit Rationale
In this unit, students will look into the fundamental concept of the derivative, which lies at the heart of calculus. They will begin by exploring the definition of a derivative as the instantaneous rate of change of a function and the slope of the tangent line at a given point. Based on their prior knowledge of algebra, trigonometry, and pre-calculus, students will learn to apply various differentiation techniques, including the product, quotient, and chain rules. The unit will also cover the differentiation of polynomial, exponential, logarithmic, and trigonometric functions. Students will engage with real-world applications of derivatives, such as motion analysis, optimization problems, and related rates, to see how these mathematical concepts translate into practical scenarios. Additionally, they will revisit and reinforce their understanding of limits and continuity, which are essential for grasping the more complex aspects of Differentiation. By the end of the unit, students will have a comprehensive understanding of calculating derivatives, interpreting their meaning, and applying them to solve various problems, thereby solidifying their foundation for further study in calculus and related fields.	The unit is a crucial part of the calculus curriculum, linking theoretical mathematics to practical applications. It builds on prior knowledge from algebra, trigonometry, and pre-calculus, reinforcing the interconnectedness of mathematical concepts. Understanding differentiation is essential for success in advanced studies and various fields such as physics, engineering, and economics. This unit enhances students' analytical and problem-solving skills, allowing them to tackle complex problems and appreciate the real-world relevance of mathematics. Additionally, it prepares students for advanced topics in calculus and promotes cognitive development by encouraging systematic problem-solving and clear reasoning. The use of technological tools further enhances their competence in solving mathematical problems. Mastering differentiation equips students with a valuable skill set for academic and professional success.

AP Standards - Calculus

MA.9-12.3.1.FUN-3.C.1	The chain rule provides a way to differentiate composite functions.
MA.9-12.3.2.FUN-3.D.1	The chain rule is the basis for implicit differentiation.
MA.9-12.3.3.FUN-3.E.1	The chain rule and definition of an inverse function can be used to find the derivative of an inverse function, provided the derivative exists.
	This topic is intended to focus on the skill of selecting an appropriate procedure for calculating derivatives. Students should be given opportunities to practice when and how to apply all learning objectives relating to calculating derivatives.
MA.9-12.3.6.FUN-3.F.1	Differentiating f' produces the second derivative f'', provided the derivative of f' exists;

	repeating this process produces higher-order derivatives of f.
MA.9-12.3.6.FUN-3.F.2	Higher-order derivatives are represented with a variety of notations. For $y = f(x)$, notations for the second derivative include d^2y/dx^2, $f''(x)$, and y''. Higher-order derivatives can be denoted d^ny/dx^n or $f^{(n)}(x)$.
MA.9-12.4.1.CHA-3.A.1	The derivative of a function can be interpreted as the instantaneous rate of change with respect to its independent variable.
MA.9-12.4.1.CHA-3.A.2	The derivative can be used to express information about rates of change in applied contexts.
MA.9-12.4.1.CHA-3.A.3	The unit for $f'(x)$ is the unit for f divided by the unit for x.
MA.9-12.4.2.CHA-3.B.1	The derivative can be used to solve rectilinear motion problems involving position, speed, velocity, and acceleration.
MA.9-12.4.3.CHA-3.C.1	The derivative can be used to solve problems involving rates of change in applied contexts.
MA.9-12.4.4.CHA-3.D.1	The chain rule is the basis for differentiating variables in a related rates problem with respect to the same independent variable.
MA.9-12.4.4.CHA-3.D.2	Other differentiation rules, such as the product rule and the quotient rule, may also be necessary to differentiate all variables with respect to the same independent variable.
MA.9-12.4.5.CHA-3.E.1	The derivative can be used to solve related rates problems; that is, finding a rate at which one quantity is changing by relating it to other quantities whose rates of change are known.
MA.9-12.4.6.CHA-3.F.1	The tangent line is the graph of a locally linear approximation of the function near the point of tangency.
MA.9-12.4.6.CHA-3.F.2	For a tangent line approximation, the function's behavior near the point of tangency may determine whether a tangent line value is an underestimate or an overestimate of the corresponding function value.
MA.9-12.4.7.LIM-4.A.2	Limits of the indeterminate forms 0/0 or ∞/∞ may be evaluated using L'Hospital's Rule.
MA.9-12.5.1.FUN-1.B.1	If a function f is continuous over the interval $[a, b]$ and differentiable over the interval (a, b), then the Mean Value Theorem guarantees a point within that open interval where the instantaneous rate of change equals the average rate of change over the interval.
MA.9-12.5.2.FUN-1.C.1	If a function f is continuous over the interval (a, b), then the Extreme Value Theorem guarantees that f has at least one minimum value and at least one maximum value on (a, b).
MA.9-12.5.2.FUN-1.C.2	A point on a function where the first derivative equals zero or fails to exist is a critical point of the function.
MA.9-12.5.2.FUN-1.C.3	All local (relative) extrema occur at critical points of a function, though not all critical points are local extrema.
MA.9-12.5.3.FUN-4.A.1	The first derivative of a function can provide information about the function and its graph, including intervals where the function is increasing or decreasing.
MA.9-12.5.4.FUN-4.A.2	The first derivative of a function can determine the location of relative (local) extrema of the function.
MA.9-12.5.5.FUN-4.A.3	Absolute (global) extrema of a function on a closed interval can only occur at critical points or at endpoints.
MA.9-12.5.6.FUN-4.A.4	The graph of a function is concave up (down) on an open interval if the function's derivative is increasing (decreasing) on that interval.
MA.9-12.5.6.FUN-4.A.5	The second derivative of a function provides information about the function and its graph, including intervals of upward or downward concavity.
MA.9-12.5.6.FUN-4.A.6	The second derivative of a function may be used to locate points of inflection for the graph of the original function.

MA.9-12.5.7.FUN-4.A.7	The second derivative of a function may determine whether a critical point is the location of a relative (local) maximum or minimum.
MA.9-12.5.7.FUN-4.A.8	When a continuous function has only one critical point on an interval on its domain and the critical point corresponds to a relative (local) extremum of the function on the interval, then that critical point also corresponds to the absolute (global) extremum of the function on the interval.
MA.9-12.5.8.FUN-4.A.9	Key features of functions and their derivatives can be identified and related to their graphical, numerical, and analytical representations.
MA.9-12.5.8.FUN-4.A.10	Graphical, numerical, and analytical information from f' and f'' can be used to predict and explain the behavior of f.
MA.9-12.5.9.FUN-4.A.11	Key features of the graphs of f, f', and $f''$are related to one another.
MA.9-12.5.10.FUN-4.B.1	The derivative can be used to solve optimization problems; that is, finding a minimum or maximum value of a function on a given interval.
MA.9-12.5.11.FUN-4.C.1	Minimum and maximum values of a function take on specific meanings in applied contexts.

Standards for Mathematical Practice

MATH.K-12.1	Make sense of problems and persevere in solving them
MATH.K-12.2	Reason abstractly and quantitatively
MATH.K-12.3	Construct viable arguments and critique the reasoning of others
MATH.K-12.4	Model with mathematics
MATH.K-12.5	Use appropriate tools strategically
MATH.K-12.6	Attend to precision
MATH.K-12.7	Look for and make use of structure
MATH.K-12.8	Look for and express regularity in repeated reasoning

Unit Focus

Enduring Understandings	Essential Questions
• Understanding that the derivative of a function represents the rate of change or the slope of the function at any given point. • Recognizing that the chain rule is essential for differentiating composite functions and is a fundamental technique in calculus. • Applying implicit differentiation to find the derivatives of functions that are not explicitly solved for one variable in terms of another. • Comprehending that the derivatives of inverse functions can be found using the properties of inverse functions and that this is	• What is the derivative, and how does it represent the rate of change of a function? • How can the chain rule be applied to differentiate composite functions? • What techniques can be used to differentiate implicitly defined functions? • How do the properties of inverse functions help in finding the derivatives of inverse functions? • In what ways do logarithmic and exponential differentiation rules differ from other

crucial in various applications. • Differentiating logarithmic and exponential functions using specific rules and understanding their applications in real-world scenarios. • Differentiating inverse trigonometric functions and understanding their significance in solving complex calculus problems. • Interpreting the meaning of a derivative in the context of real-world problems, such as motion, growth, and rates of change. • Solving problems related to speed, velocity, and acceleration using derivatives, which are fundamental concepts in physics and engineering. • Applying the concept of related rates to solve problems where multiple quantities change with respect to time. • Understanding how to evaluate limits, especially those resulting in indeterminate forms, and their importance in defining derivatives. • Applying the Mean Value Theorem to describe the behavior of functions over intervals, ensuring a deep understanding of function behavior. • Using derivatives to find local and global extrema, and applying these concepts to optimize real-world situations by maximizing or minimizing constraints.	differentiation rules? • How can we differentiate inverse trigonometric functions, and why is this important? • How can derivatives be used to interpret and solve real-world problems involving rates of change? • What role do derivatives play in solving motion problems involving speed, velocity, and acceleration? • How can we apply the concept of related rates to solve problems where multiple quantities change over time? • What strategies can we use to determine limits of functions that result in indeterminate forms? • How does the Mean Value Theorem help in understanding the behavior of functions over an interval? • What methods can be used to find local and global extrema, and how can we apply these methods to optimize real-world situations? • How do the first and second derivative tests help in analyzing and sketching the graphs of functions? • What is the significance of understanding concavity and points of inflection in the context of a function's graph? • How can the Extreme Value Theorem be applied to find extrema on intervals, and why is this useful?

Instructional Focus

Learning Targets

- Apply the chain rule.
- Use properties and rules of derivatives to differentiate symbolically.
- Find derivatives of implicitly defined functions.
- (+) Apply the properties of inverse functions to the derivatives of inverse functions.
- (+) Apply rules for bases and logarithms.
- (+) Apply rules for derivatives for inverse trig functions.
- Determine the meaning of a derivative as it relates to real-world scenarios.
- Solve motion problems involving speed, velocity and acceleration.
- Apply the derivative to solve problems involving rates of change in applied contexts.
- Calculate related rates in applied contexts.
- (+) Determine limits of functions that result in indeterminate forms
- Apply the Mean Value Theorem to describe the behavior of a function over an interval.
- Use derivatives to find local and global extrema.
- Use both open interval and closed interval methods for finding extrema.
- Use and apply the Extreme Value Theorem on intervals.
- Use the first derivative test to find intervals of increase and decrease, maximum and minimum values.
- Use the second derivative test to find points of inflection, intervals where the function is concave up or concave down, and to find maximum and minimum values.
- Use the first and second derivative tests to analyze and sketch functions.
- Optimize real-world situations by finding values that maximize or minimize constraints.

Prerequisite Skills

- Simplifying algebraic expressions.
- Solving linear and quadratic equations.
- Understanding and manipulating functions.
- Understanding the definition of a function.
- Identifying domain and range.

- Recognizing different types of functions (e.g., linear, quadratic, polynomial, exponential, logarithmic).
- Plotting points on a Cartesian coordinate system.
- Interpreting and sketching the graphs of various functions.
- Understanding the relationship between a function and its graph.
- Understanding the concept of a limit.
- Calculating limits of functions as (x) approaches a particular value.
- Recognizing and working with indeterminate forms (e.g., (\frac{0}{0})).
- Understanding what it means for a function to be continuous at a point.
- Identifying points of discontinuity in functions.
- Knowing the basic trigonometric functions (sine, cosine, tangent) and their properties.
- Understanding and using trigonometric identities.
- Graphing trigonometric functions and understanding their periodic nature.
- Understanding the properties and graphs of exponential functions.
- Understanding the properties and graphs of logarithmic functions.
- Applying rules for manipulating exponential and logarithmic expressions.
- Understanding geometric shapes and properties.
- Using geometric formulas (e.g., area, perimeter, volume).
- Understanding the concept of a derivative.
- Basic familiarity with differentiation rules and techniques.
- Applying mathematical concepts to solve problems.
- Demonstrating logical thinking and reasoning.
- Working through multi-step problems systematically.

Common Misconceptions

- Students often think the derivative of a function at a point is the same as the function's value at that point.
- Students may mistakenly apply the chain rule incorrectly by not properly differentiating the outer and

inner functions.

- Students sometimes forget to apply the chain rule when differentiating implicitly.
- Students may incorrectly assume that the derivative of an inverse function is simply the reciprocal of the original function's derivative.
- Students often mix up the rules for differentiating logarithmic and exponential functions.
- Students may struggle to connect the abstract concept of derivatives to real-world scenarios.
- Students often have difficulty setting up and solving related rates problems correctly.
- Students may think that all limits that appear to be indeterminate are undefined.
- Students might believe the Mean Value Theorem applies to all functions without considering the conditions.
- Students might confuse local extrema with global extrema.
- Students may misuse the first and second derivative tests by not correctly finding critical points or points of inflection.
- Students might not understand how to set up optimization problems correctly.

Spiraling For Mastery

Current Unit Content/Skills	Spiral Focus	Activity
• The chain rule • Symbolic differentiation • Implicit differentiation and guidelines • Higher order derivatives • (+) Derivatives of Inverse Functions • (+) Derivatives of inverse trig functions • Interpretations of the	• Skills in simplifying algebraic expressions are crucial for manipulating functions before differentiating. • Techniques for solving linear and quadratic equations are foundational when finding critical points and solving related rates problems. • Understanding the behavior of polynomial functions, including their graphs and roots, is	• iXL Diagnostic Assessment • iXL Problems • Delta Math

derivative in context
• Straight line motion
  ○ Position, velocity and acceleration
• Related rates problems
• (+) L'Hospital's Rule
• The Mean Value Theorem
• Rolle's Theorem
• Extrema on an interval
  ○ Absolute and Local Maximums and Minimums
• Extreme Value Theorem
• Define a critical number
• Increasing, Decreasing and the First Derivative Test
• Concavity and the Second Derivative test
• Sketching graphs of functions and their derivatives
• Optimization problems
	essential as students differentiate these functions.
	• Prior knowledge of different types of functions (e.g., polynomial, exponential, logarithmic) and their graphs helps students understand how derivatives affect these functions.
	• Familiarity with trigonometric functions and their properties is necessary for differentiating these functions and solving trigonometric-related problems.
	• Understanding the properties of conic sections (circles, ellipses, parabolas, and hyperbolas) can be useful when dealing with related rates and optimization problems.
	• Knowledge of geometric shapes and their properties aids in visualizing and solving optimization and related rates problems.
	• Familiarity with geometric formulas (e.g., area, volume) is often needed when setting up and solving applied differentiation problems.
	• Concepts from physics, such as velocity and acceleration, directly apply to problems involving derivatives, particularly in motion analysis.
	• Understanding how rates of change are used in physics	

	provides a real-world context for the application of derivatives. • The concept of limits is foundational for understanding derivatives. Students must be comfortable with evaluating limits to grasp the definition of the derivative. • Understanding continuity is essential for applying the Mean Value Theorem and ensuring that functions behave predictably. • Skills in logical reasoning and critical thinking are reinforced through multi-step differentiation problems and proofs. • Prior experience in setting up and solving real-world problems using mathematical models is crucial for tackling related rates and optimization problems. • Solving and graphing quadratic functions, which are often revisited when dealing with optimization problems. • Differentiation of these functions builds on students' prior knowledge from pre-calculus. • Using identities to simplify and differentiate trigonometric functions. • Techniques for simplifying complex polynomial expressions before	

	differentiation.	

Assessment

Formative Assessment	Summative Assessment
• Homework • Lesson Checks • Quizzes • Exit Tickets • Lesson Reflections • Performance Tasks	Mid Unit Assessment - Conceptual Differentiation Benchmark 2 (Linkit) AP Benchmark 2 (Linkit)

Resources

Key Resources	Supplemental Resources
Larson, R., & Edwards, B. (2010). *Calculus* (9th ed.). Brooks/Cole. • Chapters 2 and 3 Calculus Online Textbook - Openstax	iXL Delta Math Math Medic AP Classroom Desmos Activity Builder Desmos Graphing Calculator Explorations Khan Academy APSI Resources Teacher made Worksheets

Career Readiness, Life Literacies, and Key Skills

PFL.9.1.12.PB.2	Prioritize financial decisions by considering alternatives and possible consequences.
TECH.9.4.12.CT.1	Identify problem-solving strategies used in the development of an innovative product or practice (e.g., 1.1.12acc.C1b, 2.2.12.PF.3).
TECH.9.4.12.CT.2	Explain the potential benefits of collaborating to enhance critical thinking and problem solving (e.g., 1.3E.12profCR3.a).
TECH.9.4.12.TL.1	Assess digital tools based on features such as accessibility options, capacities, and utility for accomplishing a specified task (e.g., W.11-12.6.).
TECH.9.4.12.TL.4	Collaborate in online learning communities or social networks or virtual worlds to analyze and propose a resolution to a real-world problem (e.g., 7.1.AL.IPERS.6).
TECH.9.4.12.IML.1	Compare search browsers and recognize features that allow for filtering of information.
TECH.9.4.12.IML.3	Analyze data using tools and models to make valid and reliable claims, or to determine optimal design solutions (e.g., S-ID.B.6a., 8.1.12.DA.5, 7.1.IH.IPRET.8).
TECH.9.4.12.IML.7	Develop an argument to support a claim regarding a current workplace or societal/ethical issue such as climate change (e.g., NJSLSA.W1, 7.1.AL.PRSNT.4).

Interdisciplinary Connections

PFL.9.1.12.D	Planning, Saving, and Investing
PFL.9.1.12.G	Insuring and Protecting
9-12.HS-LS2	Ecosystems: Interactions, Energy, and Dynamics
9-12.HS-PS2	Motion and Stability: Forces and Interactions