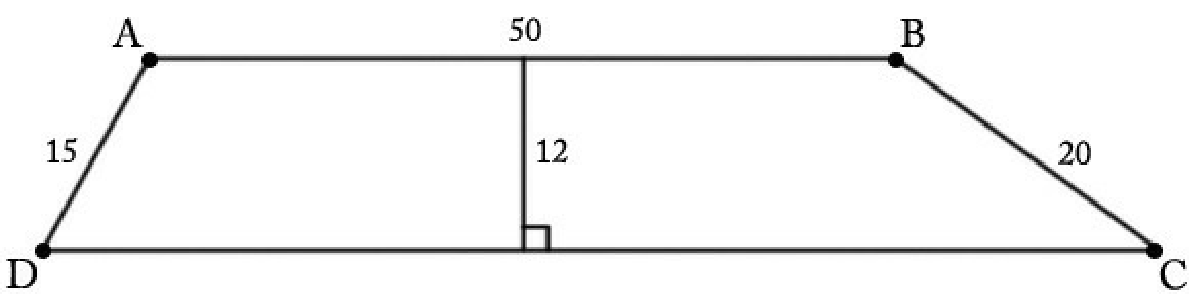
8.G Area of a Trapezoid

Alignments to Content Standards: 8.G.B.7

Task

Quadrilateral $ABCD$ is a trapezoid, $AD = 15$, $AB = 50$, $BC = 20$, and the altitude is

a. What is the area of the trapezoid?



IM Commentary

The purpose of this task is for students to use the Pythagorean Theorem to find the unknown side-lengths of a trapezoid in order to determine the area. This problem will require creativity and persistence as students must decompose the given trapezoid into other polygons in order to find its area. In the solution provided, perpendiculars from A and B are drawn to the base $\overline{DC}$ and then the Pythagorean theorem is used to find the missing segment lengths. Alternatively, $\overline{AB}$ could be extended and perpendiculars can be drawn from C and D to this extended segment, giving a rectangle from which two triangles need to be subtracted to get trapezoid $ABCD$. Finally, one of the diagonals $\overline{AC}$ or $\overline{DB}$ could be drawn, dividing $ABCD$ into two triangles and the area of these triangles can be calculated. All three methods require two applications of the Pythagorean theorem in order to complete the calculation.

Ideally, students should be thinking in terms of adding auxiliary lines (MP7), but some students may not think of such an approach. If a group of students are struggling unproductively with this task, the teacher could suggest that the students draw one or both of the altitudes shown in the solution as a way to provide some scaffolding.

This task was adapted from problem #20 on the 2011 American Mathematics Competition (AMC) 8 Test. The responses to the multiple choice answers for the problem had the following had the following distribution:

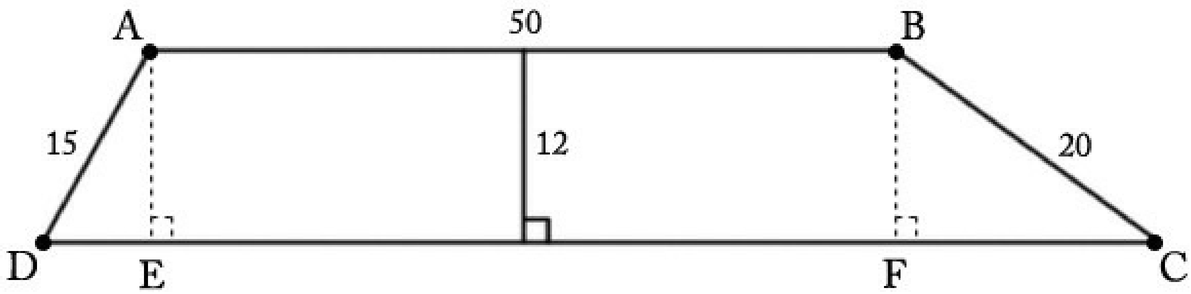
Choice	Answer	Percentage of Answers
(A)	600	10.84
(B)	650	16.80
(C)	700	18.85
(D)*	750	30.78
(E)	800	10.25
Omit	-	12.40

Of the 153,485 students who participated, 72,648 or 47% were in 8th grade, 50,433 or 33% were in 7th grade, and the remainder were less than 7th grade.

Edit this solution

Solution

We can divide the trapezoid into a rectangle and two right triangles by drawing perpendiculars from A and B to the base $\overline{DC}$ as in the picture below:



To see that $ABFE$ is a rectangle, note that since ABC is a trapezoid, this means that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel. Since $\overleftrightarrow{AE}$ and $\overleftrightarrow{BF}$ are perpendicular to $\overleftrightarrow{DC}$ they are also perpendicular to $\overleftrightarrow{AB}$. In particular, angles EAB and FBA are right angles and quadrilateral $ABFE$ is a rectangle.

We can now find the area of $ABCD$ by calculating the areas of $\triangle DAE$, $\triangle CBF$, and rectangle $ABFE$. Segements $\overline{AE}$ and $\overline{BF}$ are both altitudes for rectangle $ABCD$ and so are congruent to the segment of length 12 units given in the picture. Since angle AED is a right angle we can apply the Pythagorean theorem to find

$$|DE|^2 + |AE|^2 = |AD|^2$$

Plugging in $|AD| = 15$ and $|AE| = 12$ we find that $|DE|^2 = 225 - 144 = 81$. So $|DE| = 9$. so $|DE| = 9$. Since angle BFC is a angle we may apply the Pythagorean theorem to find

$$|FC|^2 + |BF|^2 = |BC|^2.$$

Plugging in $|BC| = 20$ and $|BF| = 12$ we find $|FC|^2 = 400 - 144 = 256$ so $|FC| = 16$.

Putting this information together, the area of $\triangle DAE$ is $\frac{1}{2} \times 9 \times 12 = 54$. The area of $\triangle CBF$ is $\frac{1}{2} \times 16 \times 12 = 96$. Finally, rectangle $ABCD$ is 50 by 12 so it has area 600. Adding these three quantities gives

$$\text{Area}(ABCD) = 54 + 96 + 600 = 750.$$

Alternatively, if students know the formula for the area of a trapezoid, namely one half the height times the sum of the lengths of the two parallel sides, we have that that the height is 12. The two parallel sides have lengths 50 and 50 + 16 + 9 or 75. So applying the formula gives

$$\frac{1}{2} \times 12 \times (50 + 75) = 750.$$