

# 8.NS Converting Repeating Decimals to Fractions

Alignments to Content Standards: 8.NS.A.1

## Task

Leanne makes the following observation:

*I know that*

$$\frac{1}{11} = 0.0909 \dots$$

*where the pattern 09 repeats forever. I also know that*

$$\frac{1}{9} = 0.1111 \dots$$

*where the pattern 11 repeats forever. I wonder if this is a coincidence?*

- What is the decimal expansion of  $\frac{1}{99}$ ? Use this to explain the patterns Leanne observes for the decimals of  $\frac{1}{9}$  and  $\frac{1}{11}$ .
- What is the decimal expansion of  $\frac{1}{999}$ ? Use this to help you calculate the decimal expansions of  $\frac{1}{27}$  and  $\frac{1}{37}$ . How does this relate to Leanne's observations?

## IM Commentary

The purpose of this task is to study some concrete examples of repeating decimals and find a way to convert them to fractions. The key observation is to use the fractions  $\frac{1}{9}$ ,  $\frac{1}{99}$ ,  $\frac{1}{999}$ , .... These fractions all have particularly simple decimal expansions which can be used to write repeating decimals as fractions.

This task leads naturally to a simple method for writing repeating decimals as fractions which the teacher may wish to explore. This technique is very closely related to the denominators 99, 999, .... Suppose we have a repeating decimal such as

$$0.137137 \dots$$

where the three digits 137 go on forever. The ideas presented in this task suggest that this decimal number is equal to  $\frac{137}{999}$ . A second, important way to see this uses an algebraic argument. If we set  $x = 0.137137 \dots$  then  $1000x = 137.137137 \dots$ . We can then perform subtraction:

$$\begin{array}{r} 137.137137 \dots \\ - 0.137137 \dots \\ \hline \end{array}$$

The result of the subtraction is 137 since the numbers after the decimal all cancel out. This shows that

$$1000x - x = 137.$$

Solving for  $x$  gives  $x = \frac{137}{999}$ .

The algebraic approach of the previous paragraph and the arithmetic approach in the solution are both important. The reason why the arithmetic approach is adapted in the solution here is that Leanne already knows what the fraction representation of these decimals are and she is interested in explaining the pattern that she observes. The explanation for the pattern turns out to be a key for writing *any* repeating decimal as a fraction.

[Edit this solution](#)

## Solution

a. The decimal expansion of  $\frac{1}{99}$  is 0.0101... where the pattern 01 repeats forever.

From the point of view of the long division algorithm, 99 does not go into 1 or into 10 but it goes into 100 with a remainder of 1. We add two more zeroes and then 99 goes in once again with a remainder of 1. This pattern goes on forever. We have

$$\begin{aligned}\frac{1}{11} &= \frac{9}{99} \\ &= 9 \times \frac{1}{99} \\ &= 9 \times 0.0101 \dots \\ &= 0.0909 \dots\end{aligned}$$

In order to justify the last step, we can look at the division algorithm. Just as for  $\frac{1}{99}$ , when we calculate  $\frac{9}{99}$  we find that 99 does not go into 9 or into 90 but it does go into 900 nine times with a remainder of 9. We will bring down two more zeroes and then 99 will again go in nine times with a remainder of 9. This gives us the decimal expansion 0.0909... above.

The same methods work for the decimal of  $\frac{1}{9}$ :

$$\begin{aligned}\frac{1}{9} &= \frac{11}{99} \\ &= 11 \times \frac{1}{99} \\ &= 11 \times 0.0101 \dots \\ &= 0.1111 \dots\end{aligned}$$

The last step in these calculations can be explained as above. When we perform long division to find  $\frac{1}{9}$ , 9 does not go into 1 but it goes into 10 once with a remainder of 1. We add a zero and 9 again goes in once with a remainder of 1. This pattern goes on forever.

b. The decimal expansion of  $\frac{1}{999}$  works just like  $\frac{1}{99}$  except that we need to add three zeroes after the decimal before 999 goes in once with a remainder of 1. We then add three more zeroes and 999 goes in once again with a remainder of 1. This pattern continues forever and so

$$\frac{1}{999} = 0.001001 \dots$$

Just as  $9 \times 11 = 99$ , the numbers 27 and 37 have been chosen because  $27 \times 37 = 999$ . Working as above we have

$$\begin{aligned}\frac{1}{27} &= \frac{37}{999} \\ &= 37 \times \frac{1}{999} \\ &= 37 \times 0.001001 \dots \\ &= 0.037037 \dots\end{aligned}$$

To justify the last step in this calculation note that 999 does not go into 37 or into 370 but it does go into 3700 three times with a remainder of 703. Then 999 goes into 7030 seven times with a remainder of 37. This is where we started and so the pattern 037 repeats forever in this decimal.

We can argue similarly for  $\frac{1}{37}$ :

$$\begin{aligned}\frac{1}{37} &= \frac{27}{999} \\ &= 27 \times \frac{1}{999} \\ &= 27 \times 0.001001 \dots \\ &= 0.027027 \dots\end{aligned}$$

The last step in the calculation is justified in the same way as for  $\frac{1}{37}$ .

Leanne noticed that  $9 \times 11 = 99$  and the decimals for the unit fractions are

$$\begin{aligned}\frac{1}{9} &= 0.1111 \dots \\ \frac{1}{11} &= 0.0909 \dots\end{aligned}$$

So the repeating part of  $\frac{1}{9}$  is 11 and the repeating part of  $\frac{1}{11}$  is 09. For  $\frac{1}{27}$  and  $\frac{1}{37}$  we have a similar phenomenon:

$$\frac{1}{27} = 0.037037 \dots$$

$$\frac{1}{37} = 0.027027 \dots$$

Again the repeating part of  $\frac{1}{27}$  is 037 and the repeating part of  $\frac{1}{37}$  is 027. Experimentation with other factorizations of 999 (or more 9's) will show the same behavior. For example,  $9999 = 99 \times 101$  and

$$\frac{1}{99} = 0.01010101 \dots$$

$$\frac{1}{101} = 0.00990099 \dots$$



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