# Unit 5: Applications of the Integral

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## Title Section

## Department of Curriculum and Instruction

**Belleville Public Schools**

**Curriculum Guide**

# Calculus AP, Unit 5

# Applications of the Integral

**Belleville Board of Education**

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## Unit Overview

- Verify solutions to differentiation equations
- Estimate solutions to differentiation equations
- Interpret the meaning of a definite integral within a problem
- Apply definite integrals to problems involving the average value of a function
- Apply definite integrals to problems involving motion
- Apply definite integrals to problems involving area, volume, and length of a curve
- Use the definite integral to solve problems in various contexts
- Analyze differential equations to obtain general and specific solutions
- Interpret, create, and solve differential equations from problems in context
- Calculate Derivatives
- Solve problems involving related rates, optimization, rectilinear motion and planar motion (BC only)
- Apply definite integrals to problems involving motion
- Apply definite integrals to problems involving area, volume, and length of a curve

Students will be able to independently use their learning to express real-world problems in the language of differential equations and generalize their knowledge of geometric concepts through the Definite Integral as well as extend their knowledge of motion in one dimension to motion in multiple dimensions and to understand and explain the power of reframing a mathematical problem in a way that makes problem-solving less difficult.

## Enduring Understanding

Students will understand that:

- The derivative has multiple interpretations and applications including those that involve instantaneous rates of change
- The definite integral of a function over an interval is a mathematical tool with many interpretations and applications involving accumulation
- Antidifferentiation is an underlying concept involved in solving separable differential equations. Solving separable differential equations involves determining a function or relation given its rate of change.
- The derivative of a function is defined as the limit of a difference quotient and can be determined using a variety of strategies
- The derivative has multiple interpretations and applications including those that involve instantaneous rates of change
- The definite integral of a function over an interval is a mathematical tool with many interpretations and applications involving accumulation

## Essential Questions

**Essential Questions are:**

1. How does the integral expand our knowledge of the world we live in?
2. Why do we need differential equations to describe the universe?
3. How can small changes in our world result in gigantic changes?
4. Why are parametric functions important in describing our world?
5. Why are polar functions important in describing our world?
6. Why have numerous mathematical representations of the world developed over time?

## Exit Skills

By the end of UNIT 5 Students will know:

- Solutions to differential equations are functions or families of functions
- Derivatives can be used to verify that a function is a solution to a given differential equation
- Slope Fields provide visual clues to the behavior of solutions to first order differential equations
- For differential equations, Euler's method provides a procedure for approximating a solution or a point on a solution curve (BC only)
- A function defined as an integral represents an accumulation of a rate of change
- The definite integral of the rate of change of a quantity over an interval gives the net change of that quantity over that interval
- The limit of an approximating Riemann sum can be interpreted as a definite integral
- The average value of a function f over an interval is equal to the definite integral divided by the width of the interval
- For a particle in rectilinear motion over an interval of time, the definite integral of velocity represents the particle's displacement over the interval of time, and the definite integral of speed represents the particle's total distance traveled over the interval of time
- Areas of certain regions in the plane can be calculated with definite integrals
- Volumes of solids with known cross sections, including discs and washers, can be calculated with definite integrals
- The length of a planar curve defined by a function can be calculated using a definite integral
- The definite integral can be used to express information about accumulation and net change in many applied contexts
- Antidifferentiation can be used to find specific solutions to differential equations with given initial conditions, including applications to motion along a line, exponential growth and decay, and logistic growth (BC only)
- Some differential equations can be solved by separation of variables
- Solutions to differential equations may be subject to domain restrictions
- Accumulation functions can be used to express particular solutions to initial value problems (IO-FTOC)
- The model for exponential growth and decay arises from the statement "The rate of change of a quantity is proportional to the size of the quantity"
- The model for logistic growth arises from the statement "The rate of change of a quantity is jointly proportional to the size of the quantity and the difference between the quantity and the carrying capacity"
- Methods for calculating derivatives of real-valued functions can be extended to vector-valued functions, parametric functions, and functions in polar coordinates
- For a curve given by a polar equation, r=f(theta), derivatives of r, x, and y with respect to theta and first and second derivatives of y with respect to x can provide information about the curve.
- Derivatives can be used to determine velocity, speed, and acceleration for a particle moving along

curves given by parametric or vector-valued functions.

- The definite integral can be used to determine displacement, distance, and position of a particle moving along a curve, given by parametric or vector-valued functions
- Areas of certain regions in the plane can be calculated with definite integrals. Areas bounded by polar curves can be calculated with definite integrals
- The length of a planar curve defined by a function or by a parametrically defined curve can be calculated using a definite integral
- The definite integral can be used to express information about accumulation and net change in many applied contexts

## New Jersey Student Learning Standards (NJSLS)

Upon completion of this section, please remove all remaining descriptions, notes, outlines, examples and/or illustrations that are not needed or used.

Please list only the content-level and cross-curricular **New Jersey Student Learning Standards** applicable to the unit. **Do not list standards that are not used in the unit.**

| | |
|---|---|
| MA.K-12.1 | Make sense of problems and persevere in solving them. |
| MA.A-SSE.A.1 | Interpret expressions that represent a quantity in terms of its context. |
| MA.K-12.2 | Reason abstractly and quantitatively. |
| MA.K-12.3 | Construct viable arguments and critique the reasoning of others. |
| MA.K-12.4 | Model with mathematics. |
| MA.F-IF.B | Interpret functions that arise in applications in terms of the context |
| MA.A-SSE.B.3 | Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression. |

| | |
|---|---|
| MA.K-12.5 | Use appropriate tools strategically. |
| MA.N-Q.A.1 | Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays. |
| MA.N-Q.A.2 | Define appropriate quantities for the purpose of descriptive modeling. |
| MA.K-12.6 | Attend to precision. |
| MA.N-Q.A.3 | Choose a level of accuracy appropriate to limitations on measurement when reporting quantities. |
| MA.F-IF.C | Analyze functions using different representations |
| MA.K-12.7 | Look for and make use of structure. |
| MA.K-12.8 | Look for and express regularity in repeated reasoning. |
| MA.F-BF.A | Build a function that models a relationship between two quantities |
| MA.G-SRT.A.2 | Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides. |
| MA.A-CED.A.1 | Create equations and inequalities in one variable and use them to solve problems. |
| MA.A-CED.A.2 | Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. |
| MA.G-SRT.C.6 | Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles. |
| MA.G-SRT.C.7 | Explain and use the relationship between the sine and cosine of complementary angles. |
| MA.A-CED.A.4 | Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. |
| MA.G-SRT.C.8 | Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems. |
| MA.F-LE.B | Interpret expressions for functions in terms of the situation they model |
| MA.G-GMD.A.2 | Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures. |
| MA.G-GMD.A.3 | Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems. |
| MA.G-GMD.B.4 | Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects. |
| MA.G-MG.A | Apply geometric concepts in modeling situations |
| MA.G-MG.A.1 | Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder). |
| MA.G-MG.A.2 | Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot). |
| MA.G-MG.A.3 | Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios). |

## Interdisciplinary Connections

| | |
|---|---|
| LA.RI.11-12.1 | Accurately cite strong and thorough textual evidence, (e.g., via discussion, written response, etc.), to support analysis of what the text says explicitly as well as inferentially, including determining where the text leaves matters uncertain. |
| LA.W.11-12.1 | Write arguments to support claims in an analysis of substantive topics or texts, using valid reasoning and relevant and sufficient evidence. |
| 9-12.HS-ETS1-4 | Use a computer simulation to model the impact of proposed solutions to a complex real-world problem with numerous criteria and constraints on interactions within and between systems relevant to the problem. |
| 9-12.HS-ETS1-3 | Evaluate a solution to a complex real-world problem based on prioritized criteria and trade-offs that account for a range of constraints, including cost, safety, reliability, and aesthetics, as well as possible social, cultural, and environmental impacts. |
| 9-12.HS-ETS1-1 | Analyze a major global challenge to specify qualitative and quantitative criteria and constraints for solutions that account for societal needs and wants. |
| 9-12.HS-ETS1-2 | Design a solution to a complex real-world problem by breaking it down into smaller, more manageable problems that can be solved through engineering. |

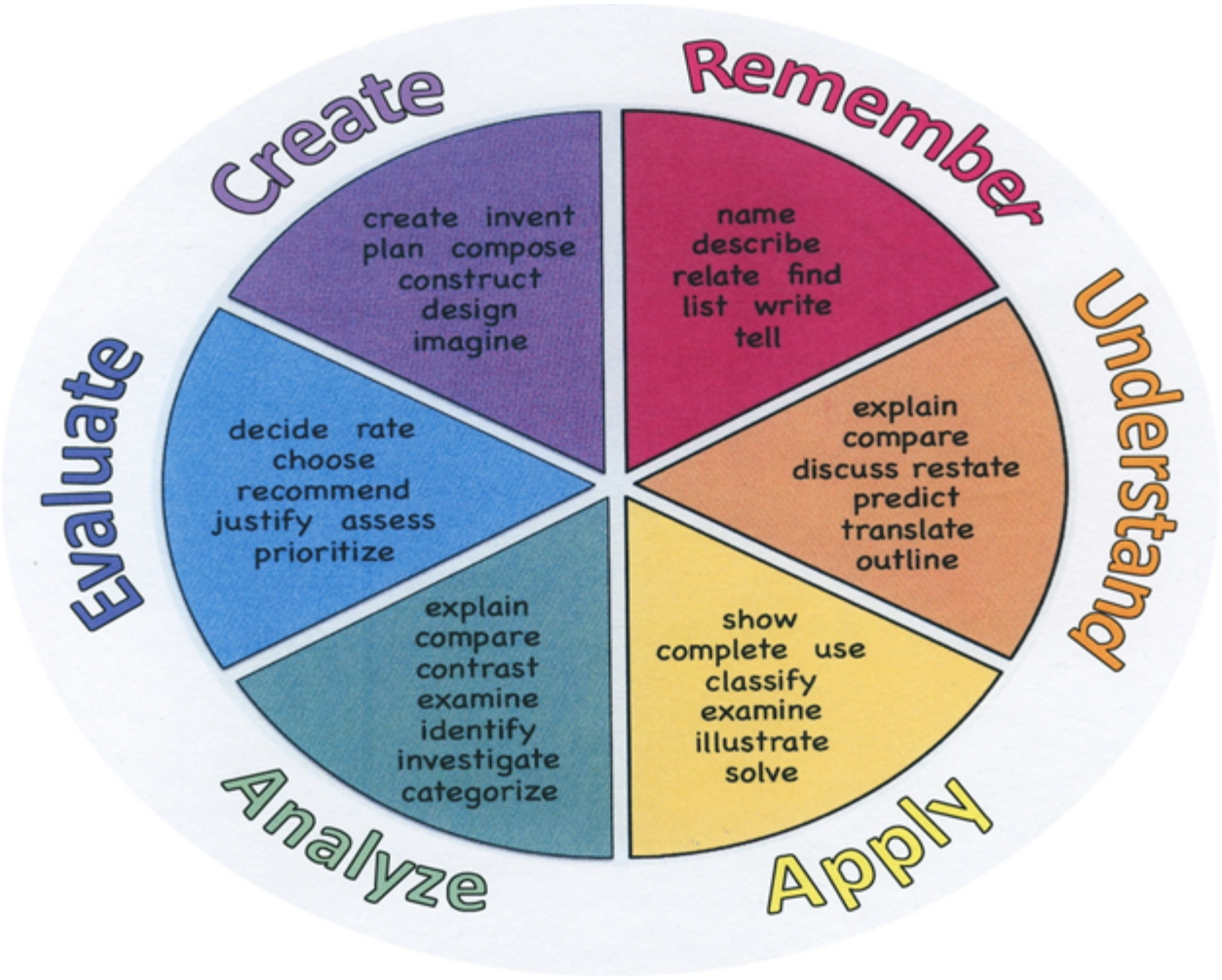
## Learning Objectives

Students will be able to:

1. Compute an area between a curve and an axis
2. Compute the area between two curves
3. Compute the arc length of a curve
4. Compute the net change of a quantity given a function describing its rate
5. Analyze motion of an object along an axis
6. Compute the average value of a function on an interval
7. Compute the volume of a solid using disk method
8. Compute the volume of a solid using washer method
9. Compute the volume of a solid using volumes of common cross-section
10. Compute areas and volume along the y-axis
11. Verify the solution to an initial value problem
12. Sketch a slope field describing an differential equation and the solution path given an initial condition
13. Use Euler's Method to approximate the solution to an initial value problem (BC only)
14. Use Separation of Variables to solve a separable differential equation or initial value problem
15. Compute velocity and acceleration vectors of a particle in planar motion
16. Compute the slope of a particle in planar motion
17. Find points of vertical or horizontal tangency of a particle in planar motion
18. Find the speed of a particle in planar motion
19. Find the change in position of a particle in planar motion
20. Find the total distance traveled by a particle in planar motion
21. Find the average speed of a particle in planar motion
22. Compute the slope of a polar curve at a point
23. Find horizontal and vertical points of tangency of a polar curve
24. Find the area enclosed by a polar curve
25. Find the area enclosed by two polar curves

**Action Verbs:** Below are examples of action verbs associated with each level of the Revised Bloom's Taxonomy.

| Remember | Understand | Apply | Analyze | Evaluate | Create |
|---|---|---|---|---|---|
| Choose | Classify | Choose | Categorize | Appraise | Combine |
| Describe | Defend | Dramatize | Classify | Judge | Compose |
| Define | Demonstrate | Explain | Compare | Criticize | Construct |
| Label | Distinguish | Generalize | Differentiate | Defend | Design |
| List | Explain | Judge | Distinguish | Compare | Develop |
| Locate | Express | Organize | Identify | Assess | Formulate |
| Match | Extend | Paint | Infer | Conclude | Hypothesize |
| Memorize | Give Examples | Prepare | Point out | Contrast | Invent |
| Name | Illustrate | Produce | Select | Critique | Make |
| Omit | Indicate | Select | Subdivide | Determine | Originate |
| Recite | Interrelate | Show | Survey | Grade | Organize |
| Select | Interpret | Sketch | Arrange | Justify | Plan |
| State | Infer | Solve | Breakdown | Measure | Produce |
| Count | Match | Use | Combine | Rank | Role Play |
| Draw | Paraphrase | Add | Detect | Rate | Drive |
| Outline | Represent | Calculate | Diagram | Support | Devise |
| Point | Restate | Change | Discriminate | Test | Generate |
| Quote | Rewrite | Classify | Illustrate | | Integrate |
| Recall | Select | Complete | Outline | | Prescribe |
| Recognize | Show | Compute | Point out | | Propose |
| Repeat | Summarize | Discover | Separate | | Reconstruct |
| Reproduce | Tell | Divide | | | Revise |
| | Translate | Examine | | | Rewrite |
| | Associate | Graph | | | Transform |
| | Compute | Interpolate | | | |
| | Convert | Manipulate | | | |
| | Discuss | Modify | | | |
| | Estimate | Operate | | | |
| | Extrapolate | Subtract | | | |
| | Generalize | | | | |
| | Predict | | | | |




## Suggested Activities & Best Practices

Upon completion of this section, please remove all remaining descriptions, notes, outlines, examples and/or illustrations that are not needed or used.

**Guidelines for Suggested Activities:**

- Includes activities **appropriate & specific** to the development of the Unit;
- Is comprised of the variety of learning activities that will be referenced in lesson plans, constructed/developed and instructionally delivered in the classroom;
- Are authentic;
- Recognizes the learning styles of the students;
- Integrates problem- or project-based learning.

## Assessment Evidence - Checking for Understanding (CFU)

- Provide open-ended problems that mirror AP Exam questions to measure comprehension (Formative)
- Peer/Self Evaluation Rubrics to measure progress (Formative)
- At the end of each chapter in the unit, summative assessments will be administered (Summative)
- Benchmark assessemnts will be administered during each quarter (Benchmark)

- Admit Tickets
- Anticipation Guide
- Common Benchmarks
- Compare & Contrast
- Create a Multimedia Poster
- DBQ's
- Define
- Describe
- Evaluate
- Evaluation rubrics
- Exit Tickets
- Explaining
- Fist- to-Five or Thumb-Ometer
- Illustration
- Journals
- KWL Chart
- Learning Center Activities
- Multimedia Reports
- Newspaper Headline
- Outline
- Question Stems
- Quickwrite

- Quizzes
- Red Light, Green Light
- Self- assessments
- Socratic Seminar
- Study Guide
- Surveys
- Teacher Observation Checklist
- Think, Pair, Share
- Think, Write, Pair, Share
- Top 10 List
- Unit review/Test prep
- Unit tests
- Web-Based Assessments
- Written Reports

## Primary Resources & Materials

Textbook: Larson, R., & Edwards, B. H. (2014). *AP Calculus 10e* (10th ed.). Independence, KY: Cengage Learning.

Texas Instruments TI-84 graphing calculator

## Ancillary Resources

KHAN Academy

https://www.khanacademy.org/math/calculus-home

Massachusetts Institute of Technology (MIT) Open Courseware for High School

https://ocw.mit.edu/high-school/mathematics/

## Technology Infusion

1. Use khanacademy.org to find instructional videos on processes of integration and finding area between curves

- iPad (for above, and YouTube math videos, as appropriate)
- Use desmos.com to find the intersection of two graphs



## Alignment to 21st Century Skills & Technology

Upon completion of this section, please remove all remaining descriptions, notes, outlines, examples and/or illustrations that are not needed or used.

Mastery and infusion of **21st Century Skills & Technology** and their Alignment to the core content areas is essential to student learning. The core content areas include:

- English Language Arts;
- Mathematics;
- Science and Scientific Inquiry (Next Generation);
- Social Studies, including American History, World History, Geography, Government and Civics, and Economics;
- World languages;
- Technology;
- Visual and Performing Arts.

## 21st Century Skills/Interdisciplinary Themes

Upon completion of this section, please remove all remaining descriptions, notes, outlines, examples and/or illustrations that are not needed or used.

Please list only the **21st Century/Interdisciplinary Themes** that will be incorporated into this unit.

- Communication and Collaboration
- Creativity and Innovation
- Critical thinking and Problem Solving
- ICT (Information, Communications and Technology) Literacy
- Information Literacy

- Life and Career Skills
- Media Literacy

## 21st Century Skills

Upon completion of this section, please remove all remaining descriptions, notes, outlines, examples and/or illustrations that are not needed or used.

Please list only the **21st Century Skills** that will be incorporated into this unit.

- Civic Literacy
- Environmental Literacy
- Financial, Economic, Business and Entrepreneurial Literacy
- Global Awareness
- Health Literacy

## Differentiation

Upon completion of this section, please remove all remaining descriptions, notes, outlines, examples and/or illustrations that are not needed or used.

Please remember: Effective educational **Differentiation** in a lesson lies within content, process, and/or product.

Please identify the ones that will be employed in this unit.

**Differentiations:**

- Small group instruction
- Small group assignments
- Extra time to complete assignments
- Pairing oral instruction with visuals
- Repeat directions
- Use manipulatives
- Center-based instruction
- Token economy
- Study guides
- Teacher reads assessments allowed
- Scheduled breaks

- Rephrase written directions
- Multisensory approaches
- Additional time
- Preview vocabulary
- Preview content & concepts
- Story guides
- Behavior management plan
- Highlight text
- Student(s) work with assigned partner
- Visual presentation
- Assistive technology
- Auditory presentations
- Large print edition
- Dictation to scribe
- Small group setting

**Hi-Prep Differentiations:**

- Alternative formative and summative assessments
- Choice boards
- Games and tournaments
- Group investigations
- Guided Reading
- Independent research and projects
- Interest groups
- Learning contracts
- Leveled rubrics
- Literature circles
- Multiple intelligence options
- Multiple texts
- Personal agendas
- Project-based learning
- Problem-based learning
- Stations/centers
- Think-Tac-Toes
- Tiered activities/assignments
- Tiered products
- Varying organizers for instructions

**Lo-Prep Differentiations**

- Choice of books or activities
- Cubing activities
- Exploration by interest
- Flexible grouping
- Goal setting with students
- Jigsaw
- Mini workshops to re-teach or extend skills
- Open-ended activities
- Think-Pair-Share
- Reading buddies
- Varied journal prompts

- Varied supplemental materials

## Special Education Learning (IEP's & 504's)

● Provide modifications and accommodations as listed in the student's IEP/504 plan
● Position student near helping peer or have quick access to teacher
● Modify or reduce assignments/texts
● Reduce length of assignment for different mode of delivery
● Break down tasks of integration to separate questions to allow students to complete one task before moving onto another
● Utilize working contract between you and student at risk
● Prioritize tasks
● Provide manipulatives
● Use graphic organizers to show various methods of integration that still arrive at the same core answer
● Use interactive math journals
● Use online resources for skill building
● Provide teacher notes on google classroom that students can print out and use in class and at home
● Use collaborative grouping strategies such as small groups
● Use online resources
● NJDOE resources

- printed copy of board work/notes provided
- additional time for skill mastery
- assistive technology
- behavior management plan
- Center-Based Instruction
- check work frequently for understanding
- computer or electronic device utilizes
- extended time on tests/ quizzes
- have student repeat directions to check for understanding
- highlighted text visual presentation
- modified assignment format
- modified test content
- modified test format
- modified test length

- multi-sensory presentation
- multiple test sessions
- preferential seating
- preview of content, concepts, and vocabulary
- Provide modifications as dictated in the student's IEP/504 plan
- reduced/shortened reading assignments
- Reduced/shortened written assignments
- secure attention before giving instruction/directions
- shortened assignments
- student working with an assigned partner
- teacher initiated weekly assignment sheet
- Use open book, study guides, test prototypes

## English Language Learning (ELL)

- Place student next to same-language speaker, if possible
- Vocabulary aides; integrals
- Use of translation dictionary or software to help explain finding area between curves
- Implement strategy groups
- Confer frequently
- Provide graphic organizers
- Modification plan
- NJDOE resources

- teaching key aspects of a topic. Eliminate nonessential information
- using videos, illustrations, pictures, and drawings to explain or clarif
- allowing products (projects, timelines, demonstrations, models, drawings, dioramas, poster boards, charts, graphs, slide shows, videos, etc.) to demonstrate student's learning;
- allowing students to correct errors (looking for understanding)
- allowing the use of note cards or open-book during testing
- decreasing the amount of workpresented or required
- having peers take notes or providing a copy of the teacher's notes
- modifying tests to reflect selected objectives
- providing study guides
- reducing or omitting lengthy outside reading assignments
- reducing the number of answer choices on a multiple choice test
- tutoring by peers
- using computer word processing spell check and grammar check features
- using true/false, matching, or fill in the blank tests in lieu of essay tests

## At Risk

● NJDOE resources
● Create weekly check-ins outside class
● Utilize online resources such as http://www.tenmarks.com or www.khanacademy.org
● Keep in contact with parents of students and track their progress, noting each communication log. Involve guidance and other teachers as necessary

- allowing students to correct errors (looking for understanding)
- teaching key aspects of a topic. Eliminate nonessential information
- allowing products (projects, timelines, demonstrations, models, drawings, dioramas, poster boards, charts, graphs, slide shows, videos, etc.) to demonstrate student's learning
- allowing students to select from given choices
- allowing the use of note cards or open-book during testing
- collaborating (general education teacher and specialist) to modify vocabulary, omit or modify items to reflect objectives for the student, eliminate sections of the test, and determine how the grade will be determined prior to giving the test.
- decreasing the amount of workpresented or required
- having peers take notes or providing a copy of the teacher's notes
- marking students' correct and acceptable work, not the mistakes
- modifying tests to reflect selected objectives
- providing study guides
- reducing or omitting lengthy outside reading assignments
- reducing the number of answer choices on a multiple choice test
- tutoring by peers
- using authentic assessments with real-life problem-solving
- using true/false, matching, or fill in the blank tests in lieu of essay tests
- using videos, illustrations, pictures, and drawings to explain or clarify

## Talented and Gifted Learning (T&G)

● Process should be modified: higher-order-thinking skills, open-ended thinking, discovery
● Utilize project-based learning for greater depth of knowledge
● Utilize exploratory connections to higher grade concepts
● Contents should be modified: abstraction, complexity, variety, organization
● Learning environments should be modified: student-centered learning, independence, openness, complexity, groups varied
● Use of web based resources such as http://www.tenmarks.com, www.khanacademy.org geogebra.org, http://www.wolframalpha.com/calculators/integral-calculator/
● NJDOE resources
● Assign projects that demonstrate real-world uses for integration to find area

- Above grade level placement option for qualified students
- Advanced problem-solving

- Allow students to work at a faster pace
- Cluster grouping
- Complete activities aligned with above grade level text using Benchmark results
- Create a blog or social media page about their unit
- Create a plan to solve an issue presented in the class or in a text
- Debate issues with research to support arguments
- Flexible skill grouping within a class or across grade level for rigor
- Higher order, critical & creative thinking skills, and discovery
- Multi-disciplinary unit and/or project
- Teacher-selected instructional strategies that are focused to provide challenge, engagement, and growth opportunities
- Utilize exploratory connections to higher-grade concepts
- Utilize project-based learning for greater depth of knowledge

## Sample Lesson