# 04- Probability, Probability Distributions, and Random Variables

Content Area: **Mathematics**
Course(s): **AP Statistics**
Time Period: **December**
Length: **9-10 blocks**
Status: **Published**

## Transfer

Exam Weighting: 10-20%

Previous Coursework: The student may have had some exposure to basic concepts of probability.

Developing Understanding: Probabilistic reasoning allows statisticians to quantify the likelihood of random events over the long run and to make statistical inferences. Simulations and concrete examples can help students to understand the abstract definitions and calculations of probability. This unit builds on understandings of simulated or empirical data distributions and fundamental principles of probability to represent, interpret, and calculate parameters for theoretical probability distributions for discrete random variables. Interpretations of probabilities and parameters associated with a probability distribution should use appropriate units and relate to the context of the situation.

Building Course Skills: Probability is a notoriously difficult topic for students to grasp because it's difficult to conceptualize future outcomes in concrete ways. Before introducing new formulas, teachers can help students get n intuitive feel for why the formulas (and related notation) make sense. For example, the probability formulas for P(A or B) and for P(A | B) can be presented intuitively with two-way tables. Simulations can also help students internalize what it means to quantify random behavior. To help students understand when to apply different probability rules, teachers can use explicit strategies such as matching verbal scenarios to their corresponding probability formulas. Students frequently misinterpret probability distributions and parameters for random variables. Teachers can reinforce that a complete interpretation will include context and units. A common misconception later in the course is that every question involving probability requires a significance test. Students should practice making predictions and decisions based on probability alone to avoid this misconception early on. They should revisit these problems in later units to practice differentiating between inference and probability problems.

Preparing for the AP Exam: To help students prepare for the AP Exam, teachers can model showing all steps in probability calculations and expect students to do the same. Calculations on the AP Exam should include presentation of an appropriate expression that communicates the structure of the formula, substitution of relevant values extracted from the problem, and an answer. In 2017 FRQ 3, for example, a student who writes "P(G) = P(G | J) • P(J) = 0.4007" has communicated the products in the multiplication rule, the sum in the addition rule, and an understanding that the events are mutually exclusive—all components of a complete response. To avoid a common error, students who present the same work using a tree diagram should practice using probabilities in the diagram correctly. Students importing incorrect solutions from one part of a multipart question to solve another will not be penalized a second time, unless the subsequent result is not a

reasonable value (like a probability less than O or greater than 1).

## Enduring Understandings

Given that variation may be random or not, conclusions are uncertain.

Simulation allows us to anticipate patterns in data.

The likelihood of a random event can be quantified.

Probability distributions may be used to model variation in populations.

Probabilistic reasoning allows us to anticipate patterns in data.

## Essential Questions

How can an event be both random and predictable?

About how many rolls of a fair six-sided die would we anticipate it taking to get three 1s?

## Student Learning Objectives

TEXT Section 5.1: Randomness, Probability, and Simulation

- Interpret probability as a long-run relative frequency
- Use simulation to model a random process.

TEXT Section 5.2: Probability Rules

- Give a probability model for a random process with equally likely outcomes and use it to find the probability of an event.
- Use basic probability rules, including the complement rule and the addition rule for mutually exclusive events.
- Use a two-way table or Venn diagram to model a random process and calculate probabilities involving two events.
- Apply the general addition rule to calculate probabilities.

TEXT Section 5.3: Conditional Independence and Probability

- Calculate and interpret conditional probabilities
- Determine if two events are independent.

- Use the general multiplication rule to calculate probabilities.
- Use a tree diagram to model a random process involving a sequences of outcomes and to calculate probabilities.
- When appropriate, use the multiplication rule for independent events to calculate probabilities.

TEXT Section 6.1: Discrete and Continuous Random Variables

- Use the probability distribution of a discrete random variable to calculate the probability of an event.
- Make a histogram to display the probability distribution of a discrete random variable and describe its shape.
- Calculate and interpret the mean (expected value) of a discrete random variable.
- Calculate and interpret the standard deviation of a discrete random variable.
- Use the probability distribution of a continuous random variable (uniform or Normal) to calculate the probability of an event.

TEXT Section 6.2: Transforming and Combining Random Variables

- Describe the effect of adding or subtracting a constant or multiplying or dividing by a constant on the probability distribution of a random variable.
- Calculate the mean and standard deviation of the sum or difference of random variables.
- Find probabilities involving sum or difference of independent Normal random variables.

TEXT Section 6.3: Binomial and Geometric Random Variables

- Determine whether the conditions for a binomial setting are met.
- Calculate and interpret probabilities involving binomial random variables.
- Calculate the mean and standard deviation of a binomial distribution and interpret these values.
- When appropriate, use the Normal approximation to the binomial distribution to calculate probabilities.
- Calculate and interpret probabilities involving geometric random variables.
- Calculate the mean and standard deviation of a geometric distribution and interpret these values.

## Vocabulary and Planned Learning Experiences

**Vocabulary:** random process, probability, law of large numbers, simulation, probability model, sample space, event, union, intersection, complement, general addition rule, mutually exclusive, two-way table, Venn diagram, conditional probability, general multiplication rule, tree diagram, independence, random variable, probability distribution, discrete random variable, continuous random variable, mean of a random variable, standard deviation of a random variable, variance, independent random variables, binomial setting, binomial random variable, binomial distribution, binomial coefficient, 10% condition, Large Counts Condition, geometric setting, geometric random variable, geometric distribution

**Planned Learning Experiences:**

**Error Analysis:** Using 2015 FRQ 3, provide students with several answers containing errors for each part. Provide some responses with incorrect notation, incorrect work, missing work, work that shows

calculator commands only, an incorrect formula or approach, and an incorrect final answer. Then ask students to identify the errors.

**Think-Pair-Share:** Provide students with a set of five probability questions: one for the complement rule, the conditional probability formula, the general multiplication rule, the multiplication rule for independent events, and the general addition rule. Ask students to individually identify the formula needed to solve each problem, without doing the final calculations. Then have them share their thoughts with a partner.

**Create Representations:** Provide students with the scenario from 2018 FRQ 3. Ask them to create a tree diagram to organize the information in the problem. Then ask them to use the information in the problem to set up a hypothetical 100,000 table (to make the decimals easy to work with), such as the one in the APCD on page. Encourage students to try both representations when solving probability questions in the future.

**Odd One Out:** After modeling an odd one out example, have students form groups of four and give each of them a description of either a binomial or a geometric random variable. Explain that three of their variables follow the same probability distribution and one is different. Have students work together in their groups to determine whose is the odd one out and explain why.

**Predict and Confirm:** Ask students to consider couples who plan to continue having children until they have one girl and predict how many children they think these couples will have, on average. Then ask each student to perform 10 trials using a coin toss where Heads = Girl and Tails = Boy. A trial is finished once one girl is observed and the number of total children is recorded. Combine the class results and calculate the average. Confirm with the geometric mean formula once it is discussed.

## Resources

TEXT: The Practice of Statistics, 6th Edition

AP Classroom and the APCD 2019 Course Description

Rossman-Chance Applets

Stats Medic

## Assessments

Reading Quiz: Randomness, Probability, and Simulation (Introductory Level)

Reading Quiz: Probability Rules (Introductory Level)

Reading Quiz: Conditional Independence and Probability (Introductory Level)

Test: Probability (AP Level)

Reading Quiz: Discrete and Continuous Random Variables (Introductory Level)

Reading Quiz: Transforming and Combining Random Variables (Introductory Level)

Reading Quiz: Binomial and Geometric Random Variables (Introductory Level)

Test: Random Variables and Probability Distributions (AP Level)

## Standards

Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not").
Understand that two events $A$ and $B$ are independent if the probability of $A$ and $B$ occurring together is the product of their probabilities, and use this characterization to determine if they are independent.
Understand the conditional probability of $A$ given $B$ as $P(A \text{ and } B)/P(B)$, and interpret independence of $A$ and $B$ as saying that the conditional probability of $A$ given $B$ is the same as the probability of $A$, and the conditional probability of $B$ given $A$ is the same as the probability of $B$.
Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities.
Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations.
Find the conditional probability of $A$ given $B$ as the fraction of $B$'s outcomes that also belong to $A$, and interpret the answer in terms of the model.
Apply the Addition Rule, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$, and interpret the answer in terms of the model.
Apply the general Multiplication Rule in a uniform probability model, $P(A \text{ and } B) = [P(A)] \times [P(B|A)] = [P(B)] \times [P(A|B)]$, and interpret the answer in terms of the model.
Use permutations and combinations to compute probabilities of compound events and solve problems.
Understand statistics as a process for making inferences about population parameters based on a random sample from that population.
Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation.
Define a random variable for a quantity of interest by assigning a numerical value to each event in a sample space; graph the corresponding probability distribution using the same graphical displays as for data distributions.
Calculate the expected value of a random variable; interpret it as the mean of the probability distribution.
Develop a probability distribution for a random variable defined for a sample space in which theoretical probabilities can be calculated; find the expected value.
Develop a probability distribution for a random variable defined for a sample space in which probabilities are assigned empirically; find the expected value.

Weigh the possible outcomes of a decision by assigning probabilities to payoff values and finding expected values.
Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator).
Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game).

| | |
|---|---|
| MA.S-IC.A.1 | Understand statistics as a process for making inferences about population parameters based on a random sample from that population. |
| MA.S-IC.A.2 | Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. |
| MA.S-CP.A.1 | Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not"). |
| MA.S-CP.A.2 | Understand that two events $A$ and $B$ are independent if the probability of $A$ and $B$ occurring together is the product of their probabilities, and use this characterization to determine if they are independent. |
| MA.S-CP.A.3 | Understand the conditional probability of $A$ given $B$ as $P(A \text{ and } B)/P(B)$, and interpret independence of $A$ and $B$ as saying that the conditional probability of $A$ given $B$ is the same as the probability of $A$, and the conditional probability of $B$ given $A$ is the same as the probability of $B$. |
| MA.S-CP.A.4 | Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities. |
| MA.S-CP.A.5 | Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. |
| MA.S-CP.B.6 | Find the conditional probability of $A$ given $B$ as the fraction of $B$'s outcomes that also belong to $A$, and interpret the answer in terms of the model. |
| MA.S-CP.B.7 | Apply the Addition Rule, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$, and interpret the answer in terms of the model. |
| MA.S-CP.B.8 | Apply the general Multiplication Rule in a uniform probability model, $P(A \text{ and } B) = [P(A)] \times [P(B\|A)] = [P(B)] \times [P(A\|B)]$, and interpret the answer in terms of the model. |
| MA.S-CP.B.9 | Use permutations and combinations to compute probabilities of compound events and solve problems. |
| MA.S-MD.A.1 | Define a random variable for a quantity of interest by assigning a numerical value to each event in a sample space; graph the corresponding probability distribution using the same graphical displays as for data distributions. |
| MA.S-MD.A.2 | Calculate the expected value of a random variable; interpret it as the mean of the probability distribution. |
| MA.S-MD.A.3 | Develop a probability distribution for a random variable defined for a sample space in which theoretical probabilities can be calculated; find the expected value. |
| MA.S-MD.A.4 | Develop a probability distribution for a random variable defined for a sample space in which probabilities are assigned empirically; find the expected value. |
| MA.S-MD.B.5 | Weigh the possible outcomes of a decision by assigning probabilities to payoff values and finding expected values. |
| MA.S-MD.B.6 | Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator). |
| MA.S-MD.B.7 | Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game). |

## Modifications (ELL, Special Education, At-Risk Students, Gifted & Talented, & 504 Plans)

ELL:

- Use visuals
- Introduce key vocabulary before lesson
- Provide peer tutoring
- Guided notes and/or scaffold outline for written assignments

Supports for Students With IEPs:

- Allow extra time to complete assignments or tests
- Guided notes and/or scaffold outline for written assignments
- Work in a small group
- Follow all IEP modifications

At-Risk Students:

- Guided notes and/or scaffold outline for written assignments
- Introduce key vocabulary before lesson
- Work in a small group
- Lesson taught again using a differentiated approach
- Use visuals / Anchor Charts

Gifted and Talented:

- Create an enhanced set of introductory activities (e.g. advance organizers, concept maps, concept puzzles)
- Organize and offer flexible small group learning activities
- Teach cognitive and methodological skills
- Organize integrated problem-solving simulations
- Propose interest-based extension activities

Supports for Students With 504 Plans:

- Follow all the 504 plan modifications
- Text to speech/audio recorded selections