Unit 5: Functions of Several Variables and Partial Derivatives

Content Area: **Mathematics**
Course(s):
Time Period: **April**
Length: **16 blocks**
Status: **Published**

Enduring Understandings

Functions of several variables should be explored and understood verbally, numerically, algebraically, and visually.

Differential calculus can be applied and related to both real and vector valued functions.

On a three dimensional surface, slope depends on direction.

Just as derivatives can be used to find tangent lines on a 2D graph, partial derivatives can be used to find tangent planes on a 3D surface.

Essential Questions

What is a contour map and what are level curves and level surfaces?

What does it mean for a function in three-space to be continuous?

How can you determine if a limit exist in three-space?

What is a partial derivative and how is it interpreted?

What is Laplace's equation and the wave equation and what are there significance?

What is a linear or tangent plane approximation of a function at a point and what is it used for?

What is a directional derivative?

What is a gradient vector and what meaning does it have?

How does one calculate the minima and maxima values of a function of several variables?

What analogies can we draw between the applications of derivatives learned in single-variable calculus and the applications of the partial derivative?

Content

Goal: The student will demonstrate the ability to use a problem-solving approach to interpret and analyze function of two

variables both graphically and algebraically, including using partial derivatives, directional derivatives, and gradients.

Instructor's Notes: Students prerequisite knowledge of differential calculus will be critically important in this unit. This is another opportunity to remediate differential calculus skills and then extend this knowledge to functions of several variables.

Instructor's Note: Partial Derivatives are one of the most important topics of multivariable Calculus. Be sure to check for understanding of this topic throughout the unit!

Vocabulary:

- **a function of two variables**
- **independent variables**
- **dependent variable**
- **Cobb-Douglas production function.**
- **level curves**
- **isobars**
- **a function of three variables**
- **half-space**
- **level surfaces**
- **a function of n variables**
- **limit of f(x,y) as (x,y) approaches (a,b)**
- **continuous at (a,b)**
- **a polynomial function of two variables**
- **a rational function of two variables**
- **partial derivative of f with respect to x at (a,b)**
- **partial derivative of f with respect to y at (a,b)**
- **second partial derivatives**
- **Clairaut's Theorem**
- **Laplace's Equation**
- **harmonic functions**
- **wave equation**
- **marginal productivity of labor**
- **marginal productivity of capital**
- **tangent plane**
- **linear approximation**
- **tangent plane approximation**
- **total differential**
- **increment of a function of three variables**
- **chain rule of a function of several variables**
- **Implicit function Theorem**
- **directional derivative**
- **gradient vector**
- **gradient**
- **tangent plane to the level surface**
- **saddle point**
- **maximum and minimum of a function of several variables**
- **critical point of a function of several variables**
- **closed set**
- **bounded set**
- **Extreme Value theorem for function of two variables**
- **Lagrange multiplier**

Skills

The student will be able to:

Define a function of two variables.

Define and determine the domain and range of a function of two variables.

Sketch the graphs of selected functions of two variables.

Define and sketch level curves and contour maps.

Test for continuity and the existence of limits of functions of two variables.

Define, compute, and interpret geometrically first and second partial derivatives for functions of more than one variable.

Define and compute the total differential for functions of more than one variable.

Apply the total differential in finding relative measurement error.

State and apply the chain rule with one or two independent variables.

Define, compute, and interpret geometrically the directional derivative of a function of two variables.

Define, compute, and interpret geometrically the gradient of a function of two variables.

State and apply the relationship between the gradient and the directional derivative for a function of two variables.

Determine the equation of a plane tangent to a surface.

Identify critical points of a surface and apply to extrema of functions of two variables.

Apply Lagrange multipliers to optimization problems.

Resources

Text Book : Stewart: *Calculus (8th edition):* Chapter 14:Partial Derivatives

Online Websites:

Kahn Academy:

https://www.khanacademy.org/math/calculus-home/multivariable-calculus

Multivariable Calculus Online by Jeff Knisley, Dept. of Math., East Tennessee State University:

http://math.etsu.edu/MultiCalc/

Online lecture notes with visuals for multivariable calculus by Paul Dawkins of Lamar University:

http://tutorial.math.lamar.edu/Classes/CalcIII/CalcIII.aspx

MIT online courseware: Calculus Revisited : Multivariable Calculus

http://ocw.mit.edu/resources/res-18-007-calculus-revisited-multivariable-calculus-fall-2011/

Drawing Space Curve Websites:

http://www.physics.ucla.edu/plasma-exp/Beam/

http://www.phy.ntnu.edu.tw/ntnujava/index.php?topic=36

Standards

Standards for Mathematical Practice

1 Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, “Does this make sense?” They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

2 Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

3 Construct viable arguments and critique the reasoning of others.

Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account

the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

4 Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5 Use appropriate tools strategically.

Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6 Attend to precision.

Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

7 Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression x2 + 9x + 14, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see 5 – 3(x – y) 2 as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

8 Look for and express regularity in repeated reasoning.

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y – 2)/(x – 1) = 3$. Noticing the regularity in the way terms cancel when expanding (x – 1)(x + 1), (x – 1)(x2 + x + 1), and (x – 1)(x3 + x2 + x + 1) might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

MA.K-12.1 Make sense of problems and persevere in solving them.

MA.K-12.2	Reason abstractly and quantitatively.
MA.K-12.3	Construct viable arguments and critique the reasoning of others.
MA.K-12.4	Model with mathematics.
MA.K-12.5	Use appropriate tools strategically.
MA.K-12.6	Attend to precision.
MA.K-12.7	Look for and make use of structure.
MA.K-12.8	Look for and express regularity in repeated reasoning.

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