Unit 4: Vector Functions

Content Area: **Mathematics**
Course(s):
Time Period: **February**
Length: **14 Blocks**
Status: **Published**

Enduring Understandings

Vector-valued functions can be used to model an object's motion through space.

Vector-valued functions can be used to describe curves and surfaces in space.

The curvature of a path can be described as the rate at which its tangent vector changes direction.

Essential Questions

What is a vector function?

What is a derivative/integral of a vector function?

What is a space curve and how do we measure its length and curvature?

What are tangent, bi-normal, and normal vectors as well as the normal and osculating planes and how do they relate to a space curve?

What is meant by the arc length parametrization"of a curve and why is this idea useful in describing curvature?

How can Kepler's Laws form physics be understood as an application of vector-valued motion through space?

What is a parametric surface and how are they defined by parametric vector functions?

Content

Goal: The student will demonstrate the ability to use a problem-solving approach to analyze vector-valued functions and use them in applications involving projectile motion and space curves.

Instructor's notes: It is important to assess students knowledge of physics at this point. Many of the applications of this unit relate to concepts in physics so differentiating the curriculum based on each individual student's understanding of physics may be necessary.

Instructor notes: There are many formulas in this unit. A decision should be made at the start of the unit with regards to which formulas students are expected to memorize versus being given. Also the unit tangent vector and normal vectors can be time consuming to solve in its entirety, so either break problem up from an intermediate step or have one problem build off the previous.

Vocabulary:

- vector-valued function
- component functions
- continuous at a vector
- space curve
- parameter
- parametric equation
- toroidal spiral
- trefoil knot
- twisted cubic
- tangent vector
- unit tangent vector
- definite integral
- parametrization
- arc length function
- parametrize a curve with respect to arc length
- smooth
- curvature
- principal unit vector
- unit normal
- bi-normal vector
- normal plane
- osculating plane
- circle of a curvature
- velocity vector
- speed
- acceleration
- Newton's Second law of Motion
- Kepler's Law of Planetray Motion

Skills

Students will be able to:

Sketch the graph of a vector valued function in two dimensions.

Determine the Cartesian equation for a two-dimensional vector-valued function.

Recognize a three dimensional vector-valued function as determining a curve in space that can be sketched as the intersection of two or more surfaces.

Differentaiate and Integrate vector-valued functions.

Apply vector-valued functions to solve problems involving velocity, acceleration, and projectile motion.

Find arc length and describe a curve using its arc length parametrization.

Calculate the unit tangent vector, curvature, and principal unit normal vector of a smooth curve.

Find the tangential and normal components of acceleration and relate this to basic physics concepts.

Define and compute curvature for a vector-values function and for a function in Cartesian form.

Resources

Text Book : Stewart: *Calculus (8th edition):* Chapter 13: Vector Functions

Online Websites:

Kahn Academy:

https://www.khanacademy.org/math/calculus-home/multivariable-calculus

Multivariable Calculus Online by Jeff Knisley, Dept. of Math., East Tennessee State University:

http://math.etsu.edu/MultiCalc/

Online lecture notes with visuals for multivariable calculus by Paul Dawkins of Lamar University:

http://tutorial.math.lamar.edu/Classes/CalcIII/CalcIII.aspx

MIT online courseware: Calculus Revisited : Multivariable Calculus

http://ocw.mit.edu/resources/res-18-007-calculus-revisited-multivariable-calculus-fall-2011/

Drawing Space Curve Websites:

http://www.physics.ucla.edu/plasma-exp/Beam/

http://www.phy.ntnu.edu.tw/ntnujava/index.php?topic=36

Standards

Standards for Mathematical Practice

1 Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

2 Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

3 Construct viable arguments and critique the reasoning of others.

Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account

the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

4 Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5 Use appropriate tools strategically.

Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6 Attend to precision.

Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

7 Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression x2 + 9x + 14, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see 5 –

3(x – y) 2 as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

8 Look for and express regularity in repeated reasoning.

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation (y – 2)/(x – 1) = 3. Noticing the regularity in the way terms cancel when expanding (x – 1)(x + 1), (x – 1)(x2 + x + 1), and (x – 1)(x3 + x2 + x + 1) might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

MA.9-12.II	Derivatives
MA.9-12.II.A	Concept of the derivative
MA.9-12.II.A.1	Derivative presented graphically, numerically, and analytically
MA.9-12.II.A.2	Derivative interpreted as an instantaneous rate of change
MA.9-12.II.A.3	Derivative defined as the limit of the difference quotient
MA.9-12.II.D	Second derivatives
MA.9-12.II.D.1	Corresponding characteristics of the graphs of f, f', and f"
MA.9-12.III	Integrals
MA.9-12.III.B	Applications of integrals
MA.9-12.III.C	Fundamental Theorem of Calculus
MA.9-12.III.C.1	Use of the Fundamental Theorem to evaluate definite integrals
MA.9-12.III.D	Techniques of antidifferentiation
MA.9-12.III.D.1	Antiderivatives following directly from derivatives of basic functions
MA.9-12.III.D.2	Antiderivatives by substitution of variables (including change of limits for definite integrals), parts, and simple partial fractions (nonrepeating linear factors only)
MA.9-12.III.E.1	Finding specific antiderivatives using initial conditions, including applications to motion along a line
MA.K-12.1	Make sense of problems and persevere in solving them.
MA.K-12.2	Reason abstractly and quantitatively.
MA.K-12.3	Construct viable arguments and critique the reasoning of others.
MA.K-12.4	Model with mathematics.
MA.K-12.5	Use appropriate tools strategically.
MA.K-12.6	Attend to precision.
MA.K-12.7	Look for and make use of structure.
MA.K-12.8	Look for and express regularity in repeated reasoning.
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relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

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