Unit 3: Vectors and Three Dimensional Space

Content Area: **Mathematics**
Course(s):
Time Period: **November**
Length: **20 blocks**
Status: **Published**

Enduring Understandings

Dot products and cross products can be powerful tools for expressing geometrical ideas.

Three dimensional surfaces can be visualized from their equations and by extending and applying our knowledge of traditional functions graphed in a plane.

Essential Questions

How are the two ways of calculating both a dot product and cross product related to each other?

What are the applications of products and cross products in the fields of geometry and physics?

As we explore multivariable calculus, what are some of the classic 3D shapes we will be examining?

Content

Goal: The student will demonstrate the ability to use a problem solving approach to apply operations on vectors in two and three dimensions and use vectors to analyze planes, cylinders, and quadric surfaces.

Instructor's notes: Students prerequisite knowledge of vectors, parametric equations, matrices and conics should be preassessed before starting this unit. Time will either be saved or added time needed to remediate these topics as student need requires.

Vocabulary:

- coordinate axes
- right-hand rule
- coordinate planes
- octants
- first octant
- three-dimensional rectangular coordinate system
- vector
- scalar
- Parallelogram Law
- scalar multiplication
- components
- position vector
- magnitude (length) of a vector

- resultant force
- dot product
- angle between two vectors
- orthogonal
- direction angles
- vector projection
- cross product
- deteminant
- scalar triple product
- vector triple product
- torque
- vector equation
- parameter
- parametric equations
- symmetric equations
- direction numbers
- normal vector
- unit vector
- cylinder
- quadric surfaces
- ellipsoid
- elliptic paraboloid
- hyperboloid of one sheet
- ruled surface

Skills

The student will be able to:

Express vectors in component form and standard basis form.

Perform basic operations of addition, subtarction and scalar multiplication on vectors, both geometrically and algebraically

Compute a unit vector i the direction of a given vector.

Express a vector as a linear combination of two other vectors.

Find the length and midpoint of a line segement or vector in three dimensions.

Calculate dot products and cross products of vectors and interpret geometrically.

Compute the angle between two vectors.

Determine whether two vectors are orthogonal.

Determine and apply the vector projection of one vector along another.

Interpret and graph points in the 3-dimensional coordinate system.

Use vectors to define the equations of lines and planes.

Write the equation of a sphere.

Determine the equation of a plane in three-dimensional coordinate system and sketch the plane.

Calculate the distance between a point and a plane.

Create all 4 conic sections.

Graph parabolas given various characteristics.

Find the standard equations of a parabola.

Determine the equation of a circle given its radius and center.

Graphs circles given their equation.

Find the equation of an ellipse given various characteristics.

Graph ellipses in standard form using center, major and minor axes.

Determine the equation of a hyperbola.

Graph hyperbolas in standard form using its center, vertices, and equations of asymptotes.

Complete the square in order to identify a conic section.

Identify and sketch the graphs of cylinders and basic types of quadric surfaces.

Recognize surfaces of revolution and their equation.

* The italicized skills may not need to be covered but simply reviewed if conics are covered in a previous course curriculum.

Resources

Text Book : Stewart: *Calculus (8th edition):* Chapter 12: Vectors and the Geometry of Space

Online Websites:

Kahn Academy:

https://www.khanacademy.org/math/calculus-home/multivariable-calculus

Multivariable Calculus Online by Jeff Knisley, Dept. of Math., East Tennessee State University:

http://math.etsu.edu/MultiCalc/

Online lecture notes with visuals for multivariable calculus by Paul Dawkins of Lamar University:

http://tutorial.math.lamar.edu/Classes/CalcIII/CalcIII.aspx

MIT online courseware: Calculus Revisited : Multivariable Calculus

http://ocw.mit.edu/resources/res-18-007-calculus-revisited-multivariable-calculus-fall-2011/

Standards

Vector and Matrix Quantities **N -VM**

A. Represent and model with vector quantities.

1. (+) Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g., v, |v|, ||v||, v).

2. (+) Find the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point.

3. (+) Solve problems involving velocity and other quantities that can be represented by vectors.

B. Perform operations on vectors.

4. (+) Add and subtract vectors.

a. Add vectors end-to-end, component-wise, and by the parallelogram rule. Understand that the magnitude of a sum of two vectors is typically not the sum of the magnitudes.

b. Given two vectors in magnitude and direction form, determine the magnitude and direction of their sum.

c. Understand vector subtraction v – w as v + (–w), where –w is the additive inverse of w, with the same magnitude as w and pointing in the opposite direction. Represent vector subtraction graphically by connecting the tips in the appropriate order, and perform vector subtraction component-wise.

5. (+) Multiply a vector by a scalar.

a. Represent scalar multiplication graphically by scaling vectors and possibly reversing their direction; perform scalar multiplication component-wise, e.g., as c(vx, vy) = (cvx, cvy).

b. Compute the magnitude of a scalar multiple cv using ||cv|| = |c|v. Compute the direction of cv knowing that when |c|v ≠0, the direction of cv is either along v (for c > 0) or against v (for c < 0).

C. Perform operations on matrices and use matrices in applications.

7. Multiply matrices by scalars to produce new matrices, e.g., as when all of the payoffs in a game are doubled

11. (+) Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors.

12. Work with 2 × 2 matrices as a transformations of the plane, and interpret the absolute value of the determinant in terms of area..

Expressing Geometric Properties with Equations G-GPE

A. Translate between the geometric description and the equation for a conic section

1. Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

2. Derive the equation of a parabola given a focus and directrix.

3. (+)Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant.

Standards for Mathematical Practice

1 Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, “Does this make sense?” They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

2 Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

3 Construct viable arguments and critique the reasoning of others.

Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account

the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

4 Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5 Use appropriate tools strategically.

Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6 Attend to precision.

Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

7 Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 – 3(x – y) 2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

8 Look for and express regularity in repeated reasoning.

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y – 2)/(x – 1) = 3$. Noticing the regularity in the way terms cancel when expanding (x – 1)(x + 1), (x – 1)(x2 + x + 1), and (x – 1)(x3 + x2 + x + 1) might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

MA.K-12.1	Make sense of problems and persevere in solving them.
MA.K-12.2	Reason abstractly and quantitatively.
MA.K-12.3	Construct viable arguments and critique the reasoning of others.
MA.K-12.4	Model with mathematics.
MA.K-12.5	Use appropriate tools strategically.
MA.K-12.6	Attend to precision.
MA.K-12.7	Look for and make use of structure.
MA.K-12.8	Look for and express regularity in repeated reasoning.
MA.N-VM	Vector and Matrix Quantities
MA.N-VM.A	Represent and model with vector quantities.
MA.N-VM.A.1	Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g., $\boldsymbol{v}$, $\|\boldsymbol{v}\|$, $\|\|\boldsymbol{v}\|\|$, $\boldsymbol{v}$).
MA.N-VM.A.3	Solve problems involving velocity and other quantities that can be represented by vectors.
MA.N-VM.B	Perform operations on vectors.
MA.N-VM.B.4	Add and subtract vectors.
MA.N-VM.B.5	Multiply a vector by a scalar.
MA.N-VM.B.4a	Add vectors end-to-end, component-wise, and by the parallelogram rule. Understand that the magnitude of a sum of two vectors is typically not the sum of the magnitudes.
MA.N-VM.B.4b	Given two vectors in magnitude and direction form, determine the magnitude and direction of their sum.
MA.N-VM.B.4c	Understand vector subtraction $\boldsymbol{v} - w$ as $\boldsymbol{v} + (-w)$, where $-w$ is the additive inverse of w, with the same magnitude as w and pointing in the opposite direction. Represent vector subtraction graphically by connecting the tips in the appropriate order, and perform vector subtraction component-wise.
MA.N-VM.B.5a	Represent scalar multiplication graphically by scaling vectors and possibly reversing their direction; perform scalar multiplication component-wise, e.g., as $c(\boldsymbol{v}_x$, $\boldsymbol{v}$ subscript y) = ($c\boldsymbol{v}_x$, $c\boldsymbol{v}$ subscript y).
MA.N-VM.B.5b	Compute the magnitude of a scalar multiple $c\boldsymbol{v}$ using $\|\|c\boldsymbol{v}\|\| = \|c\|\boldsymbol{v}$. Compute the direction of $c\boldsymbol{v}$ knowing that when $\|c\|\boldsymbol{v} \neq 0$, the direction of $c\boldsymbol{v}$ is either along $\boldsymbol{v}$ (for $c > 0$) or against $\boldsymbol{v}$ (for $c < 0$).
MA.N-VM.C	Perform operations on matrices and use matrices in applications.
MA.N-VM.C.7	Multiply matrices by scalars to produce new matrices, e.g., as when all of the payoffs in a game are doubled.
MA.N-VM.C.11	Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors.
MA.N-VM.C.12	Work with 2 × 2 matrices as a transformations of the plane, and interpret the absolute value of the determinant in terms of area.
MA.G-GPE	Expressing Geometric Properties with Equations
MA.G-GPE.A	Translate between the geometric description and the equation for a conic section

MA.G-GPE.A.1	Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.
MA.G-GPE.A.2	Derive the equation of a parabola given a focus and directrix.
MA.G-GPE.A.3	Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant.

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